UNIVERSITY OF OXFORD

THESIS

Étale homotopy sections of algebraic varieties

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ABSTRACT

We define and study the fundamental pro-finite 2-groupoid $\Pi_2(X)$ of varieties X defined over a field k. This is a higher algebraic invariant of a scheme X, analogous to the higher fundamental path 2-groupoids as defined for topological spaces. This invariant is related to previously defined invariants, for example the absolute Galois group of a field, and Grothendieck's étale π_1.

The special case of Brauer-Severi varieties X is considered, in which case a "sections conjecture" type theorem is proved. It is shown that a Brauer-Severi variety X has a rational point if and only if its étale fundamental 2-groupoid has a special sort of section.

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Introduction

Given a topological space X, Henri Poincaré defined the fundamental group $\pi_1(X,x)$ of X based at some point x of X. This is a very useful algebraic invariant of a space, and its consideration was the first example of a transformation (functor) of type:

$$\text{topology} \to \text{algebra}$$

that defines the field of algebraic topology. It aids in the translation of geometric problems into algebraic ones. For example Brouwer's fixed point theorem, which states that every continuous mapping from a disk to itself has a fixed point, becomes a simple calculation involving the existence of a commutative diagram of finitely generated groups.

The fundamental group is the second in a sequence of algebraic structures

$$\pi_0(X), \pi_1(X,x), \pi_2(X,x), \ldots$$

that describe X in algebraic terms. As it turns out, for "nice" spaces (path connected, locally path connected and locally simply connected) the fundamental group of X depends only on a category defined using X: the category of covering spaces of X, or equivalently, the category $L_0(X)$ of locally constant sheaves on X. This is because $L_0(X)$ is a "Galois category" (as defined in SGA1, [16]) with Galois group $\pi_1(X,x)$ in the sense that $\pi_1(X,x)$ "classifies" the objects of $L_0(X)$. To be precise, there is an equivalence of categories:

$$L_0(X) \simeq \mathrm{Rep}(\pi_1(X,x))$$

between $L_0(X)$ and the representations of $\pi_1(X,x)$ in **Set**, *i.e.* sets equipped with an action by $\pi_1(X,x)$ and equivariant maps.

Using these ideas, Alexander Grothendieck defined the étale fundamental group $\pi_1^{\text{ét}}(X,x)$ of a scheme X (x is now some geometric point of X). He defined it as the "Galois group" of a certain "Galois category". In this setting the fundamental group classifies the finite étale covers of the scheme X. The category of finite étale covers of X can also be described as the locally constant sheaves on X, for the étale site.

Once this algebraic invariant of schemes was set up, he was able to state the *sections conjecture*, a conjecture relating algebraic information encoded in $\pi_1(X,x)$ and rational points of an *anabelian* scheme X. See Grothendieck's letter to Faltings [17] for the first statement of this conjecture. In this letter Grothendieck explains that there are schemes that belong to some (loosely) defined class An, whose fundamental groups are non-commutative. These should include, in particular, the smooth proper, geometrically connected and hyperbolic algebraic curves over some field k, that is finitely generated over $\mathbf{Q}$.

More precisely, given an algebraic closure $\bar{k}$ of k, there is a short exact sequence on pro-finite groups:

$$\mathbf{1} \longrightarrow \pi_2^{\text{ét}}(\bar{X},\bar{x}) \longrightarrow \pi_1^{\text{ét}}(X,x) \xrightarrow{\text{st}} G_k \longrightarrow \mathbf{1}$$

where st is the image of the structure morphism $X \to \mathrm{Spec}(k)$, and x is a geometric point of X. Let $\mathrm{Sec}(\pi_1^{\text{ét}}(X))$ denote the set of splittings of this short exact sequence, modulo conjugacy.

Sections Conjecture. *Let X be an anabelian variety. If k is finitely generated over $\mathbf{Q}$, then the natural mapping $X(k) \to \mathrm{Sec}(\pi_1^{ét}(X))$ is a bijection.*

The goal is thus the same as for topology: to translate algebraic-geometry problems, like the existence of rational points, to purely algebraic ones, like the existence of splittings of a short exact sequence of pro-finite groups.

One can ask:

Question. *Can the section conjecture (and other "anabelian conjectures") be generalised in any way to a class of schemes strictly larger than* An*?*

By only keeping track of the algebraic data contained within $\pi_1^{\text{ét}}(X,x)$ there is little hope; indeed An is somewhat circularly defined to be maximal for these sorts of properties to hold. What happens though, if an algebraic model of X that is more refined that $\pi_1^{\text{ét}}(X,x)$ is used?

A first question is to ask how to define $\pi_2(X)$ of a scheme, an algebraic structure that would be analogous to the second fundamental group of a topological space. One can try to mimic the previous definition, that is, to find a category of things defined over X that $\pi_2(X)$ is meant to classify. This turns out however to be misguided since the natural thing to classify is now the 2-category $L_1(X)$ of locally constant stacks on X, and these are not encoded by $\pi_2(X)$ for the trivial reason that if you want to classify locally constant stacks you are also going to need to classify locally constant sets (since sets can be construed as special sorts of groupoids) since the latter are part of the former. The algebraic object needed should be a "π_2" that still contains the information held in π_1.

Higher category theory provides such an object, $\Pi_2(X)$, and the inclusion

$$L_0(X) \rightarrowtail L_1(X)$$

is reflected by a *decategorification*

$$\Pi_2(X) \to \Pi_1(X).$$

This is the *fundamental 2-groupoid* of X, and in [4] Waschkies and Polesello indeed prove that (for X a topological space) it classifies the category $L_1(X)$ of locally constant stacks on X:

$$L_1(X) \simeq \mathrm{Rep}(\Pi_2(X)) := [\Pi_2(X), \mathbf{Gpd}].$$

This new algebraic invariant of X, $\Pi_2(X)$, is a 2-category, which means that there are not only objects and morphisms, but also 2-morphisms between the morphisms. Roughly, $\Pi_2(X)$ can be defined as follows:

- objects are the points of X,
- 1-morphisms are the paths in X,
- 2-morphisms are equivalence classes (under higher homotopy) of homotopies between paths. "Vertical composition" is concatenation of homotopies, and "horizontal composition" is like Godement composition of functors.

The story does not end there, there is a whole sequence of n-groupoids, with $\Pi_n(X)$ classifying locally constant $(n-1)$-stacks on X. However the foundations are harder to deal with.

The idea is then to define an analogous object $\Pi_n^{\text{ét}}(X)$ for a scheme X, with $\Pi_2^{\text{ét}}(X)$ containing in particular the data of Grothendieck's $\pi_1^{\text{ét}}(X,x)$, and to ask questions about "higher sections conjectures" involving these invariants.

Artin and Mazur defined and developed an étale homotopy theory of a scheme X in [13]. The object $h(X)$ they define is a pro-object in the homotopy category of simplicial sets:

$$h(X) \in \text{pro}(\text{Ho}(\mathbf{SSet})).$$

Later, Friedlander (in [21]) defined the notion of an étale topological type in the category of pro-simplicial sets that induces Artin and Mazur's étale homotopy type in the pro-homotopy category.

In [18], Toën and Vezzosi define a similar pro-object, this time however indexed itself by a higher-category (rather than a category as in the case of Artin-Mazur and Friedlander). The construction is more natural and generalises Toën's previous work in *"Vers une interpretation Galoisienne de la théorie de l'homotopie"* [20], where he proves that locally constant ∞-stacks on a (suitable) topological space are classified by functors from $\Pi_\infty(X)$ into $\infty\mathbf{Gpd}$. In the last section of [18], the authors give a sketch of a proof of this in the case of a Π_∞ defined for a general ∞-topos, and hint that their construction probably generalises that of Artin and Mazur for schemes.

Here, we have decided to restrict our study to $\Pi_2(X)$, trying to define this object directly using the theory of locally constant stacks on X, but our construction follows the ideas that are presented in [18]. It is our hope that this smaller and more manageable object would be more amenable to direct calculations and study, than say Artin and Mazur's $h(X)$.

After discussing the construction of $\Pi_2(X)$ in the case of schemes, a type of higher sections conjecture is studied for varieties that are not in An. These are the Severi-Brauer varieties. In this case the following theorem is proved.

0.1. **Theorem.** *A Severi-Brauer variety X over a field k has a k-rational point if and only if the structure morphism of the corresponding étale fundamental pro-2-groupoids*

$$\Pi_2^{\acute{e}t}(X) \to \Pi_2^{\acute{e}t}(\mathrm{Spec}(k)) \simeq \mathbf{B}\mathrm{Gal}(\bar{k}|k)$$

has a weak pseudo-geometric section.

We hope that this will encourage number theorists to consider $\Pi_2^{\text{ét}}(X)$ as a worthy generalisation of Galois groups, and an algebraic object worth studying for number-theoretic interests.

Structure of the thesis

In part 1, we recall some of the definitions we need from higher category theory and present the definition of pro-2-groupoids. For pro-finite-2-groupoids, a representation theorem, similar to the Yoneda lemma, is proved.

In part 2, we recall the basic definitions of stacks and locally constant stacks, since $\Pi_2(X)$ will be defined by the universal property that it classifies these objects.

In part 4, we present a generalisation of the classical "category of elements" construction, which is used in the proof that $\Pi_2(X)$ exists. This is essentially the construction of the category that will index the pro-object $\Pi_2(X)$. The main theorem is that any functor $F\colon C^{\mathrm{op}} \to 2\mathbf{Gpd}$, where C is a 3-category, is a colimit of representable functors. In other words, there is a pro-object that co-represents F.

In part 5 we give the definition and proof of existence of $\Pi_2(X)$. The fact that it classifies locally constant finite stacks on $X_{\text{ét}}$ is shown. We calculate $\Pi_2(X)$ when X is a field, and show that it is equivalent to the delooping of the absolute Galois group of X. We then study some comparison theorems for $X_K \to X$, when X is a scheme over a field k and $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ an extension of fields. We extend the classification result so that we can also classify locally constant A-gerbes on $X_{\text{ét}}$, where A is a sheaf of abelian groups on $\mathrm{Spec}(k)_{\text{ét}}$.

Since the construction of $\Pi_2(X)$ is functorial in X, for any morphism of schemes $f\colon X \to Y$, there is a morphism of pro-finite 2-groupoids $\Pi_2(f)\colon \Pi_2(X) \to \Pi_2(Y)$. In part 6 we give a condition on morphisms of pro-finite 2-groupoids $\varphi\colon \Pi_2(X) \to \Pi_2(Y)$, that makes it more likely that φ comes from an actual morphism of schemes.

In part 7 we explain how the structures that we have defined can be used to state conjectures and study rational points on varieties.

Part 8 studies the special case of $\Pi_2(X)$ when X is a Severi-Brauer variety. In this case we prove that a pseudo-geometric section of the structure morphism $\Pi_2(X) \to \mathbf{B}G_k$ does give rise to the existence of a rational point. This is done by using a pseudo-geometric section s as a sort of stabiliser, allowing us to descend a trivialisation of a Chern class representing a certain gerbe on X. This then proves that a certain line bundle is rational, and hence that a rational point exists.

In part 9, we discuss how the notion of a pseudo-geometric notion might fit into a higher theory, which is a more complete way of “preserving geometric structure” at the level of homotopy types.

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Conventions.

- All fields considered will be of characteristic 0.
- The ring of integers is denoted **Z**.
- The field of rational numbers is denoted **Q**.

Part 1. **Categories**

1. 3-Categories

1.1. The material below uses the theory of higher categories. Higher categories are a generalisation of categories where there are objects, morphisms between the objects, and morphisms between the morphisms, and so on. Defining precisely what is meant by this has proved difficult, and has even developed into a field of mathematics in its own right. An introduction by John Baez to these ideas can be found in [1].

In [2], Baez and Dolan give a definition of weak n-categories based on opetopes and opetopic sets. This definition was later refined by various people, including Claudio Hermida, Michael Makkai and John Power. See for example [10].

In this text, we will only need 3-categories, and the various structures between them (functors, natural transformations, etc.). These have been carefully defined using a more traditional approach in [3], as bicategory enriched categories. These are essentially categories $\mathcal{C}$ in which for each pair of objects A and B, there is a bicategory of morphisms $\mathcal{C}(A,B)$.

1.2. Definition. A 3-*category* is a tricategory as in [3]. See definition 3.1.1 of [3]. We will also use the definition of functors, natural transformations, modifications, etc. as given in section 4 of [3]. The structure given by 3-categories and these transformations is referred to as 3**Cat**.

1.3. All 3-categories considered here will be of the *weak* (not strict) sort, and similarly for functors between them. Furthermore, all limits and colimits, unless otherwise specified, will be 3-limits, that is, limits/colimits taken in a 3-category.

1.4. Definition. A 2-*category* is a 3-category such that all 3-morphism hom-sets are singleton. There is an equivalence (of 3-categories) between 2-categories, as just defined, and the classical definition of weak 2-categories, for example, bicategories. Similar definitions apply to 1-categories and 0-categories. The 3-category of 2-categories is denoted 2**Cat**.

1.5. Definition. A *n-groupoid* (for $0 \leqslant n \leqslant 3$) is an n-category such that all i-morphisms ($1 \leqslant i \leqslant n$) are equivalences. The $(n+1)$-category of n-groupoids is denoted n**Gpd**.

1.6. Remark. Note that the $(n+1)$-category n**Gpd** has the following property: all the i-morphisms, for $i \geqslant 1$, are equivalences. This is because the 2-morphisms (and above) are defined by 1-morphisms in the n-groupoids under consideration, which are themselves equivalences. Such categories are called $(n, 1)$-categories and are easier to manipulate than general n-categories.

1.7. Notation. There are inclusion functors

$$\mathbf{Set} \simeq 0\mathbf{Gpd} \rightarrowtail 1\mathbf{Gpd} \rightarrowtail 2\mathbf{Gpd} \rightarrowtail 3\mathbf{Gpd}$$

whose notation will often be suppressed. Thus, given a 1-groupoid M and a 3-groupoid N, we will freely speak of the 3-groupoid of functors $M \to N$.

1.8. Notation. The symbol **1** denotes the terminal object in whatever category is under consideration. Mostly it will refer to the trivial i-groupoid ($0 \leqslant i \leqslant 3$), which is the unique groupoid (up to equivalence) with a unique k-morphism for all k, $0 \leqslant k \leqslant i$.

1.9. Notation. When defining a higher category explicitly by describing the i-morphisms, we will use a numbered list, with the item $\{i\}$ describing the i-morphisms. Note that the composition for i-morphisms is defined, following Baez-Dolan, at the next level. For example, here is an explicit definition of **BZ**:

$\{0\}$ $\{*\}$

$\{1\}$ $\{a \colon * \to * \mid a \in \mathbf{Z}\}$

$\{2\}$ $\{a \circ b \to c \mid a + b = c\}$

This should be read as follows:

{0} "There is a unique object named $*$"

{1} "For each element a of $\mathbf{Z}$ there is a morphism $a\colon * \to *$ names a"

{2} "The composite of a and b is $a + b$", or more conceptually: "There is a unique 2-morphism $a \circ b \to c$ exactly when $a + b = c$"

1.10. Notation. Given two categories M and N, the category of functors $M \to N$ will be denoted N^M, or $[M, N]$. Thus for example, the Yoneda embedding for a 2-groupoid M is in $[M, [M^{\mathrm{op}}, \mathbf{Gpd}]]$.

1.11. Definition. A *finite* i-category ($0 \leqslant i \leqslant 3$) is an i-category with a finite set of i-morphisms.

1.12. Definition. The 3-category of finite n-groupoids is denoted $\mathbf{Fin}n\mathbf{Gpd}$.

2. The Yoneda lemma

2.1. There is a proof of the Yoneda lemma for *cubical* 3-categories. A definition of cubical 3-categories and the proof of the Yoneda lemma for such 3-categories can be found in [3]. We will only use the Yoneda lemma for the 3-categories $2\mathbf{Gpd}$ and $\mathbf{Fin}2\mathbf{Gpd}$, which are cubical.

2.2. Lemma. *For any cubical 3-category $\mathcal{C}$, there is a functor*

$$Y\colon \mathcal{C} \to 3\mathbf{Cat}(\mathcal{C}^{\mathrm{op}}, 2\mathbf{Cat})$$

that is an equivalence of 3-categories of $\mathcal{C}$ onto the sub-3-category of $3\mathbf{Cat}(\mathcal{C}^{\mathrm{op}}, 2\mathbf{Cat})$ of representable functors.

2.3. Let M be a 2-groupoid. Denote by $[-, M]$ the functor

$$2\mathbf{Gpd}^{\mathrm{op}} \to 2\mathbf{Gpd}$$

that M represents. For any two functors $F, G\colon 2\mathbf{Gpd}^{\mathrm{op}} \to 2\mathbf{Gpd}$, the 2-category of morphisms between them will be denoted $[F, G]$.

2.4. Let M be a 2-groupoid and $F\colon \mathbf{2Gpd}^{\mathrm{op}} \to \mathbf{2Gpd}$ a functor, then by the Yoneda lemma for 3-categories applied to $\mathbf{2Gpd}$, there is a natural equivalence

$$[[-,M],F] \simeq F(M).$$

In particular, if M and N are two 2-groupoids, there is a natural equivalence

$$[[-,M],[-,N]] \simeq \mathbf{2Gpd}(M,N).$$

2.5. Similarly, the Yoneda lemma applied to the 3-category $\mathbf{2Gpd}^{\mathrm{op}}$ gives a natural equivalence, for any two 2-groupoids M and N:

$$[[M,-],[N,-]] \simeq \mathbf{2Gpd}(N,M).$$

This will be used in lemma 5.4 below, which is a sort of co-Yoneda lemma for pro-finite-2-groupoids.

3. Limits and colimits of 2-groupoids

We recall here the basic definitions for limits of 2-groupoids. The definitions and proofs for colimits are very similar to those of limits. The existence and theory of limits for $\mathbf{2Gpd}$ is much easier than a general 3-category.

3.1. Definition. Let D be a 3-category, and let $F\colon D^{\mathrm{op}} \to \mathbf{2Gpd}$ be a functor. Define the functor

$$\hat{\varprojlim}(F)\colon \mathbf{2Gpd}^{\mathrm{op}} \to \mathbf{2Gpd}$$

by

$$\hat{\varprojlim}(F)(T) := [D^{\mathrm{op}}, \mathbf{2Gpd}](\mathbf{1}, \mathbf{2Gpd}(T, F(-)).$$

That is, $\hat{\varprojlim}(F)$ takes the 2-groupoid T to the 2-groupoid of natural transformations

$$\mathbf{1} \to \mathbf{2Gpd}(T, F(-))$$

where

- $\mathbf{1}$ is the terminal functor $D^{\mathrm{op}} \to 2\mathbf{Gpd}$,
- $2\mathbf{Gpd}(T, F(-))$ is the functor

$$D^{\mathrm{op}} \to 2\mathbf{Gpd}$$
$$d \mapsto 2\mathbf{Gpd}(T, F(d)),$$

  that maps a groupoid d to the 2-groupoid of functors $T \to F(d)$.

3.2. Definition. Let D be a 3-category and $F\colon D^{\mathrm{op}} \to 2\mathbf{Gpd}$ a functor. A *limit* of F is a 2-groupoid L that represents $\hat{\varprojlim}(F)$.

3.3. Thus if L is the limit of $F\colon D^{\mathrm{op}} \to \mathbf{Gpd}$, then for any 2-groupoid T, the functors

$$f\colon T \to L$$

are equivalent to natural transformations

$$\mathbf{1} \to 2\mathbf{Gpd}(T, F(-)).$$

Such a natural transformation associates

- to any $d \in \mathcal{D}$, a functor
$$f(d)\colon T \to F(d),$$
- to any $d_{12}\colon d_1 \to d_2$ in D, an equivalence
$$f(d_{12})\colon F(d_{12}) \circ f(d_2) \to f(d_1),$$
- etc.

3.4. Lemma. *The limit of a functor* $F\colon D^{\mathrm{op}} \to 2\mathbf{Gpd}$ *is, if it exists, unique up to equivalence.*

Proof. Suppose two 2-groupoids are a limit of F. Then the functors $2\mathbf{Gpd}^{\mathrm{op}} \to 2\mathbf{Gpd}$ that they represent are both equivalent to $\hat{\varprojlim} F$, and they are thus equivalent. Hence by the Yoneda lemma, the two groupoids must be equivalent. □

3.5. Lemma. *Limits of 2-groupoids exist.*

Proof. An easy calculation shows that the limit of $F\colon D^{\mathrm{op}} \to 2\mathbf{Gpd}$ is given by the following 2-groupoid:

$$[D^{\mathrm{op}}, 2\mathbf{Gpd}](\mathbf{1}, F),$$

that is, the 2-groupoid of natural transformations between the terminal functor $\mathbf{1}\colon D^{\mathrm{op}} \to 2\mathbf{Gpd}$ and F. □

3.6. Lemma. Given a 3-category D, forming the limit over functor $F\colon D^{\mathrm{op}} \to 2\mathbf{Gpd}$ is natural in F, that is, produces a functor

$$\varprojlim_D \colon [D^{\mathrm{op}}, 2\mathbf{Gpd}] \to 2\mathbf{Gpd}.$$

Proof. The functor $\varprojlim_D$ is given by co-represented by $\mathbf{1}$ in $[D^{\mathrm{op}}, 2\mathbf{Gpd}]$. □

4. PRO-2-GROUPOIDS

We will need to use pro-2-groupoids in some of the constructions below. These are pro-objects in the category of 2-groupoids. The main difference from pro-objects in the usual 1-categorical sense, apart from taking values in a higher category, is that they can also be indexed by higher categories.

4.1. Definition. Let C be a 3-category. A *diagram* in C is a functor $A \to C$ for some 3-category A. A *finite diagram* is a diagram $A \to C$ where A is a finite category.

4.2. Definition. Let C be a 3-category and $f\colon A \to C$ a diagram in C. A *cone* on f is an object c of C and a morphism $\Delta(c) \to f$, where $\Delta(c)$ is the constant functor $A \to C$ with value c.

4.3. Definition. A 3-category C is (finitely) *cofiltered* if every finite diagram in C has a cone.

4.4. Definition. The *category of pro-2-groupoids*, denoted pro2**Gpd**, is defined as the following 3-category:

$\{0\}$ The objects are diagrams $F\colon D \to 2\mathbf{Gpd}$, where D is a small cofiltered 3-category.

{1,2,3} Given objects $F\colon D \to 2\mathbf{Gpd}$ and $G\colon E \to 2\mathbf{Gpd}$, the 2-category of morphisms $F \to G$ is defined by the following limit-colimit construction. Consider the functor

$$E \times D^{\mathrm{op}} \to 2\mathbf{Gpd}$$

$$(e,d) \mapsto 2\mathbf{Gpd}(F(d),G(e))$$

Or, by Currying,

$$E \to [D^{\mathrm{op}}, 2\mathbf{Gpd}].$$

Composing this with the functor $\varinjlim_{D^{\mathrm{op}}}$ gives:

$$E \longrightarrow 2\mathbf{Gpd}^{D^{\mathrm{op}}} \xrightarrow{\varinjlim_{D^{\mathrm{op}}}} 2\mathbf{Gpd},$$

a functor over which we may now take the limit. The result is the 2-groupoid of morphisms between F and G. We will write this colimit-limit as

$$\varprojlim_{e:E} \varinjlim_{d:D^{\mathrm{op}}} 2\mathbf{Gpd}(F(d),G(e)).$$

4.5. Before continuing with the definition, we describe some of the data included in a morphism of pro-2-groupoids. Let $A\colon \mathcal{D}_A \to 2\mathbf{Gpd}$ and $B\colon \mathcal{D}_B \to 2\mathbf{Gpd}$ be two pro-objects. Then a morphism $f\colon A \to B$ can be described by the following data (by virtue of being an object of a filtered limit of 2-groupoids):

- For each $b \in \mathcal{D}_B$, some object

$$f(b) \in \varinjlim_{a \in \mathcal{D}_A^{\mathrm{op}}} 2\mathbf{Gpd}(A(a),B(b)).$$

- The object $f(b)$ can in turn be described by a pair $(a(b), f(b,a(b)))$ where $a(b) \in \mathcal{D}_A$, and $f(b,a)$ is a functor

$$A(a(b)) \to B(b)$$

- The structure of the limit-colimit adds coherence morphisms to $f(b,a)$. For example for each morphism $b_{12}\colon b_1 \to b_2$ in $\mathcal{D}_B$, there must be some $a_{12}\colon a(b_1) \to a(b_2)$ and some (weakly invertible) natural transformation:

$$b_{12} \circ f(b_1, a(b_1)) \to f(b_2, a(b_2)) \circ a_{12}.$$

And similarly the 2-morphisms in $\mathcal{D}_B$ add coherence structure to these, and so on.

4.6. To complete the definition, we need to show the existence of composition. We sketch the proof of this. We need to define composition of morphisms in pro2**Gpd**. Given pro-2-groupoids:

$$A\colon \mathcal{D}_A \to 2\mathbf{Gpd}$$

$$B\colon \mathcal{D}_B \to 2\mathbf{Gpd}$$

$$C\colon \mathcal{D}_C \to 2\mathbf{Gpd}$$

and morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

We must define gf as an object of

$$\varprojlim_{c:\mathcal{D}_C} \varinjlim_{a:\mathcal{D}_A^{\mathrm{op}}} 2\mathbf{Gpd}(A(a), C(c)).$$

Defining data for gf is straightforward, we show here the first two steps:

- For each $c \in \mathcal{D}_C$,
  - There is some $b(c) \in \mathcal{D}_B$ and a morphism

$$g(c)\colon B(b(c)) \to C(c),$$

  - There is, for $b(c) \in \mathcal{D}_B$ some $a(b(c))$ in $\mathcal{D}_A$ and a morphism

$$f(b(c))\colon A(a(b(c))) \to B(b(c)),$$

- Composing gives us an object $a(b(c))$ of $\mathcal{D}_A$ and a morphism

$$gf(c) := \quad A(a(b(c))) \xrightarrow{f(b(b))} B(b(c)) \xrightarrow{g(c)} C(c).$$

- For each morphism $c_{12} \colon c_1 \to c_2$ in $\mathcal{D}_C$,
  - There is some morphism $b(c_{12}) \colon b(c_1) \to b(c_2)$ and some $g(c_{12})$ that is an equivalence:

$$\begin{array}{ccc} B(b(c_1)) & \xrightarrow{g(c_1)} & C(c_1) \\ {\scriptstyle B(b(c_{12}))}\big\downarrow & \overset{g(c_{12})}{\Swarrow} & \big\downarrow{\scriptstyle C(c_{12})} \\ B(b(c_2)) & \xrightarrow[g(c_2)]{} & C(c_2) \end{array}$$

  - There is some $a(b(c_{12})) \colon a(b(c_1)) \to a(b(c_2))$ and some equivalence $f(b(c_{12}))$:

$$\begin{array}{ccc} A(a(b(c_1))) & \xrightarrow{f(b(c_1))} & B(b(c_1)) \\ {\scriptstyle A(a(b(c_{12})))}\big\downarrow & \overset{f(b(c_{12}))}{\Swarrow} & \big\downarrow{\scriptstyle C(c_{12})} \\ A(a(b(c_2))) & \xrightarrow[f(b(c_2))]{} & B(b(c_2)) \end{array}$$

  - These compose to the required equivalence $gf(c_{12})$:

$$\begin{array}{ccccc} A(a(b(c_1))) & \xrightarrow{f(b(c_1))} & B(b(c_1)) & \xrightarrow{g(c_1)} & C(c_1) \\ {\scriptstyle A(a(b(c_{12})))}\big\downarrow & \overset{f(b(c_{12}))}{\Swarrow} & \big\downarrow{\scriptstyle B(b(c_{12}))} & \underset{g(c_{12})}{\Swarrow} & \big\downarrow{\scriptstyle C(c_{12})} \\ A(a(b(c_2))) & \xrightarrow[f(b(c_2))]{} & B(b(c_2)) & \xrightarrow[g(c_2)]{} & C(c_2). \end{array}$$

One then has to show that the data thus obtained gives a well defined element of

$$\underset{c:\mathcal{D}_C}{\underleftarrow{\lim}} \ \underset{a:\mathcal{D}_A^{\mathrm{op}}}{\underrightarrow{\lim}} \ 2\mathbf{Gpd}(A(a), C(c)).$$

Most of the work is done for us by the lower morphisms. It's only for 2-morphisms of $\mathcal{D}_C$ that one has to check that the representative system of 3-morphisms picked in

$$\varinjlim_{a:\mathcal{D}_A^{\mathrm{op}}} 2\mathbf{Gpd}(A(a), C(c))$$

obeys some coherence laws with respect to 3-morphisms in $\mathcal{D}_C$.

5. Pro-finite-2-groupoids

5.1. Definition. A *pro-finite-2-groupoid* is a pro-object in the 3-category of finite 2-groupoids.

Thus a pro-finite-2-groupoid is a pro-2-groupoid $M\colon \mathcal{D}_M \to 2\mathbf{Gpd}$ such that for all $d \in \mathcal{D}_M$, $M(d)$ is a finite 2-groupoid.

5.2. Let M and N be 2-groupoids, and consider the functors $[M,-]$ and $[N,-]$:

$$2\mathbf{Gpd} \to 2\mathbf{Gpd}$$

that are co-represented by M and N respectively. Then recall that the (co)Yoneda lemma states in particular that

$$[[M,-],[N,-]] \simeq [N,M]$$

that is, there is an equivalence between the morphisms of functors

$$[M,-] \to [N,-]$$

and the morphisms of 2-groupoids $M \to N$. In the following we prove a similar statement regarding pro-finite-2-groupoids.

5.3. Let $M\colon \mathcal{D}_M \to \mathbf{Fin}2\mathbf{Gpd}$ be a pro-finite-2-groupoid. Then we can define the functor $\mathbf{Fin}2\mathbf{Gpd} \to 2\mathbf{Gpd}$ that it co-represents:

$$[M,-]\colon \mathbf{Fin}2\mathbf{Gpd} \to 2\mathbf{Gpd}$$
$$T \mapsto \mathrm{pro}2\mathbf{Gpd}(M,T).$$

Here, pro2**Gpd**(M,N) is the category of morphisms of pro-2-groupoids $M \to N$ where N is considered a constant pro-2-groupoid. Again, morphisms $[M,-] \to [N,-]$ are just natural transformations. In this way we have extended the natural Yoneda embedding

$$\mathbf{Fin2Gpd}^{\mathrm{op}} \to [\mathbf{Fin2Gpd}, \mathbf{Fin2Gpd}]$$

to a functor

$$\mathrm{pro}\mathbf{Fin2Gpd}^{\mathrm{op}} \to [\mathbf{Fin2Gpd}, \mathbf{2Gpd}].$$

The goal is to show that this extension also satisfies the Yoneda property.

5.4. Lemma. *Let M and N be two pro-finite-2-groupoids. Then there is an equivalence of categories*

$$[[M,-],[N,-]] \simeq [N,M]$$

Proof. Let $f\colon [M,-] \to [N,-]$ be a natural transformation. In the Yoneda lemma we would proceed to extract the M-component of f, however this does not exist here. However for each $d \in \mathcal{D}_M$ we may take the $M(d)$-component of f:

$$f(M(d))\colon [M,M(d)] \to [N,M(d)],$$

which is natural in d. There is a natural morphism

$$i_d\colon M \to M(d)$$

which is simply the projection of M onto its component $M(d)$. Evaluating $f(M(d))$ at i_d produces

$$g_d := f(M(d))(i_d)\colon N \to M(d).$$

The morphism $g_d\colon N \to M(d)$ is represented by an object of

$$\varinjlim_{e \in \mathcal{D}_N^{\mathrm{op}}} [N(e), M(d)].$$

Thus taken as a whole we get a system $(g_d \mid d \in \mathcal{D}_M)$ that represents some object of

$$\varprojlim_{d \in \mathcal{D}_M} \varinjlim_{e \in \mathcal{D}_N} [N(e), M(d)],$$

that is, by the definition of a morphism of pro-2-groupoids, a morphism $N \to M$. It is straightforward to check that this transformation is natural in f, that is, produces a functor

$$[[M,-],[N,-]] \simeq [N,M].$$

The inverse transformation simply takes f to the natural transformation $[M,-] \to [N,-]$ that is pre-composition with f at each component. $\square$

Part 2. Stacks and locally constant objects

5.5. A definition of $\Pi_2(X)$ can be obtained if we have an understanding of the appropriate locally constant objects that live over X.

Studying the locally constant functions on a topological space X informs us of the connectivity in X: a locally constant function must be constant over each connected component. Thus a locally constant function $X \to S$ (to a set S) can be presented by giving an element $a \in S$ for each connected component of X. Thus the set of locally constant functions with values in S is equivalent to the set of functions

$$\Pi_0(X) \to S.$$

A locally constant sheaf $\mathcal{F}$ with values in a 1-category $\mathcal{V}$ on a topological space X has to be constant over a cover $(U_i \to U \mid i \in I)$. We can thus present $\mathcal{F}$, after restricting the cover to simply connected connected pieces, by giving, for each i in I, an object V_i, and some coherence conditions:

- over each double intersection $U_{ij} = U_i \times_X U_j$, an isomorphism $f_{ij} \colon V_i \to V_j$,
- such that, over each triple intersection $U_i \times_X U_j \times_X U_k$, $g_i k = g_{jk} \circ g_{ij}$.

In this case, the locally constant $\mathcal{V}$-valued sheaves are equivalent to functors

$$\Pi_1(X) \to \mathcal{V}.$$

In our case we need the *locally constant stacks*. Being two levels higher than locally constant functions, this will inform us of connectivity two levels higher than the connected components: how paths are connected, and how paths-of-paths are connected. Thus, it will inform us of what the homotopy 2-type is.

This was done for topological spaces by Polesello and Waschkies in [12] for locally relatively 2- connected topological spaces. They defined a "monodromy 2-functor", and using this to establish, for a 2-category $\mathcal{V}$, an equivalence between the $\mathcal{V}$-valued locally constant stacks on X, and the functor category $[\Pi_2(X), \mathcal{V}]$ (see Theorem 2.2.5 in [12]).

We give the definitions that we need here, but refer the reader to [7] and [6] for details on 2-stacks.

6. Stacks

For the whole of this section, S will denote a site.

6.1. In this section we review the theory of stacks. Let $\mathcal{V}$ be a 2-category with all finite limits. This will be the category of *values* for our stack. A typical choice for $\mathcal{V}$ is **Gpd**, the 2-category of groupoids.

6.2. Definition. A *pre-stack* on S with values in $\mathcal{V}$ (or, a $\mathcal{V}$*-prestack*) is a functor

$$S^{\mathrm{op}} \to \mathcal{V}.$$

More generally, the 2-category of $\mathcal{V}$-prestacks on S is the functor category $[S^{\mathrm{op}}, \mathcal{V}]$. Denote this 2-category by $\mathrm{PSt}_S(\mathcal{V})$.

6.3. Put simply, a stack (on S with values in $\mathcal{V}$) is a prestack that preserves certain limits. This is the basic idea of a sheaf/stack: you can construct the sections over an object U by limiting over the values of the stack over a cover of U.

6.4. Definition. A prestack P on S with values in $\mathcal{V}$ is a *stack* if P commutes with filtered 2-limits indexed by coverings that are stable by finite intersections. The full subcategory of $\mathrm{PSt}_S(\mathcal{V})$ whose objects are stacks is denoted $\mathrm{St}_S(\mathcal{V})$.

6.5. Explanation. Consider a diagram $d\colon I \to S$ in the site S, where the family $(d(i) \mid i \in I)$ is a covering family for the site S of some object U. Assuming that S is stable under finite intersection means that for any diagram

$$\begin{array}{ccc} i_1 & & i_2 \\ & {\scriptstyle \iota_1}\searrow \quad \swarrow{\scriptstyle \iota_2} & \\ & U & \end{array}$$

in I, there is some $j \in I$ such that $d(j)$ is equivalent to the limit of

$$d(i_1) \xrightarrow{\iota_1} U \xleftarrow{\iota_2} d(i_2)$$

in S.

Assuming that a diagram is filtered in S^{op} is the same as assuming that the opposite diagram is cofiltered in S. The diagram d is cofiltered if I is cofiltered, that is, if each finite sub-diagram of I has a cone.

Because S is a site, the colimit of d in S is U.

Thus the assumption above for some pre-stack P asks that for each diagram d as just described,

$$P(\varprojlim(\ I^{\mathrm{op}} \xrightarrow{d} S^{\mathrm{op}} \ )) \simeq P(U) \simeq \varprojlim(\ I^{\mathrm{op}} \xrightarrow{d} S^{\mathrm{op}} \xrightarrow{F} \mathcal{V} \ ).$$

6.6. Sheaves. We will explain how the above definition restricts to the definition of sheaves when we consider **Set** as the value category. Let F be some pre-stack (of sets).

Let d be a diagram as in the previous sub-section. Then

$$\varprojlim(\ I^{\mathrm{op}} \xrightarrow{d} S^{\mathrm{op}} \xrightarrow{F} \mathcal{V} \ )$$

can be simplified as follows. Because this is a cofiltered limit, the elements of this limit are families

$$(s_i \in F(i) \mid i \in I)$$

such that for every morphism $i_{12} \colon i_1 \to i_2$ in I

$$F(i_{12})(s_{i_2}) = s_2.$$

Such families are called threads. Given any element s of $F(U)$, we get a thread by restriction:

$$(s|_{d(i)} \mid i \in I).$$

A priori the condition above is stronger than the sheaf condition, since it asks that elements of $F(U)$ be threads, rather than families of elements

$$(s_i \in F(d(i)) \mid i \in I)$$

that agree on restriction to intersections. However this isn't the case because triple intersections don't matter in this case. Indeed for any triple intersection of $U_1, U_2, U_3 \to U$, the diagram (for example)

$$\begin{array}{ccccc} & & F(U_{123}) & & \\ & \swarrow & & \searrow & \\ F(U_{12}) & & & & F(U_{12}) \\ & \searrow & & \swarrow & \\ & & F(U_1) & & \end{array}$$

is required to commute strictly (since **Set** is a 1-category), and so checking compatibility at triple intersections becomes superfluous.

6.7. Covering limit diagrams of interest. Just like the condition for a presheaf (of sets) to be a sheaf only relies on double-intersections, the definition of a stack of groupoids only relies on triple intersections. Here is a more explicit description of how this works.

Consider a covering family $(f_i \colon U_i \to U \mid i \in I)$. For each pair $(i, j) \in I^2$ there is a comma object:

$$\begin{array}{ccc} (f_i/f_j) & \xrightarrow{p_{ij}} & U_i \\ {\scriptstyle q_{ij}}\downarrow & & \downarrow{\scriptstyle f_i} \\ U_j & \xrightarrow[f_j]{} & U, \end{array}$$

which is just the fibre product. For each triple $(i, j, k) \in I^3$ we have double comma objects $(f_i/f_j/f_k)$, which is equivalent to:

$$(f_i/f_j) \times_{U_j} (f_j/f_k)$$

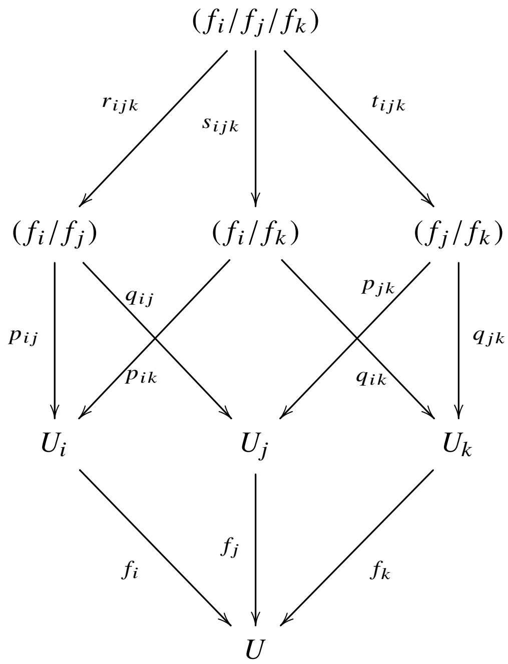
for all other permutations of i, j, k. This provides the following projections:

$$r_{ijk} : (f_i/f_j/f_k) \to (f_i/f_j),$$

$$s_{ijk} : (f_i/f_j/f_k) \to (f_j/f_k),$$

$$t_{ijk} : (f_i/f_j/f_k) \to (f_i/f_k).$$

This is all summarised in the following cubical diagram, in which each node is the limit of all the nodes it is a source of.



Let $\mathcal{D}(f_i/f_j/f_k)$ denote the above diagram in S, and $\mathcal{L}(f_i/f_j/f_k)$ the same diagram but with the vertex U omitted.

The following lemma gives an alternate definition of a stack to the one above. It is equivalent to the definition given in [6].

6.8. Lemma. *Let $\mathcal{F} \colon S^{\mathrm{op}} \to$ **Gpd** be a prestack of groupoids. For each covering family $(f_i : U_i \to U \mid i \in I)$, and each triple $(i,j,k) \in I^3$, consider $\mathcal{F}(\mathcal{D}(f_i/f_j/f_k))$, the application of $\mathcal{F}$ to $\mathcal{D}(f_i/f_j/f_k)$, which is thus a diagram in $\mathcal{V}$. We obtain a cone on the diagram $\mathcal{F}(\mathcal{L}(f_i/f_j/f_k))$*

with vertex $\mathcal{F}(U)$. The prestack $\mathcal{F}$ is a stack if, for each covering family $(f_i \mid i \in I)$ and each triple (i, j, k), $\mathcal{F}(\mathcal{D}(f_i/f_j/f_k))$ exhibits $\mathcal{F}(U)$ as the limit of $\mathcal{F}(\mathcal{L}(f_i/f_j/f_k))$.

7. The 2-stack of $\mathcal{V}$-valued stacks

7.1. We recall here the construction of the 2-stack of stacks with values in a 2-category $\mathcal{V}$ that has all small limits. For details on this construction, see Appendix B in [12], which itself also refers to [7].

7.2. For any object U of S, we may consider the 2-category of $\mathcal{V}$-valued stacks on U. Given $U_1 \to U_2$ in S, $\mathcal{V}$-valued stacks on U_2 restrict to $\mathcal{V}$-values stacks on U_1. This process makes the system of $\mathcal{V}$-valued stacks on U a pre-2-stack on the site S. Denote this pre-2-stack by $\mathrm{St}_{\mathcal{V}}(S)$.

7.3. Lemma. *The pre-2-stack* $\mathrm{St}_{\mathcal{V}}(S)$ *is a 2-stack on S.*

7.4. There is no way, for a general 2-category $\mathcal{V}$, to construct a stackification functor that associates to any pre-stack a corresponding stack. However for stacks with values in **Gpd**, which we will be considering below (for example Gerbes), this is possible. For general $\mathcal{V}$, we will be giving a more general definition of “locally constant $\mathcal{V}$-valued stack”.

7.5. Definition. A **Gpd**-valued stack $\mathcal{F}$ is *constant* if it is the stackification of a constant prestack. A **Gpd**-valued stack $\mathcal{F}$ is locally constant if there is a cover $(U_i \mid i \in I)$ in S such that the restrictions $\mathcal{F}|_{U_i}$ are constant stacks, for all $i \in I$.

8. Locally constant $\mathcal{V}$-valued stacks

8.1. Locally constant objects over a space / site follow a hierarchical pattern. This goes as follows:

- A locally constant function is a section of a constant sheaf;
- A locally constant sheaf is a section of a constant stack;
- A locally constant stack is a section of a locally constant 2-stack;
- etc.

8.2. Let P be a pre-2-stack. A 2-stack F equipped with a morphism $f\colon P \to F$ is a *stackification* of P if the following universal property is satisfied: for any stack T and any morphism $g\colon P \to T$, g must factor through f up to a unique equivalence:

$$\begin{array}{ccc} P & \xrightarrow{g} & T \\ {\scriptstyle f}\big\downarrow & \nearrow{\scriptstyle \exists!} & \\ F. & & \end{array}$$

This and more is summarised by the following proposition.

8.3. Proposition. *The forgetful functor:*

$$F\colon \mathrm{St}_2(S) \to \mathrm{PSt}_2(S)$$

has a left adjoint functor

$$\#\colon \mathrm{PSt}_2(X) \to \mathrm{St}_2(X).$$

Proof. See Proposition 2.0.3 in [12]. □

8.4. Definition. The functor # is called *stackification*.

8.5. Definition. A *constant pre-2-stack* $F\colon S^{\mathrm{op}} \to \mathcal{V}$ is a pre-2-stack that is constant as a functor. Sending a 2-category $\mathcal{V}$ to its corresponding constant pre-2-stack is a functor:

$$\Delta\colon \mathcal{V} \to \mathrm{PSt}_2(S).$$

8.6. Definition. A *constant 2-stack* is the stackification of a constant pre-2-stack. If $\mathcal{V}$ is a 2-category, then the stackification of $\Delta(\mathcal{V})$ is a constant 2-stack and is referred to as the *constant 2-stack with stalk* $\mathcal{V}$.

8.7. The composite functor

$$\mathrm{const}_S\colon\ 2\mathbf{Cat} \xrightarrow{\quad\Delta\quad} \mathrm{PSt}_2(S) \xrightarrow{\#} \mathrm{St}_2(S)$$

that sends a 2-category $\mathcal{V}$ to the constant 2-stack with stalk $\mathcal{V}$ is denoted const_S. Recall that

$$\Gamma\colon \mathrm{St}_2(S) \to 2\mathbf{Cat}$$

denotes the global sections functor.

8.8. Definition. Let $\mathcal{V}$ be a 2-category with all small limits. An object of $\Gamma(\mathrm{const}_S(\mathcal{V}))$ is called a *$\mathcal{V}$-valued locally constant stack on S. More generally, the 2-stack* $\mathrm{const}_S(\mathcal{V})$ *is the 2-stack of locally constant $\mathcal{V}$-valued stacks on S.*

8.9. Notation. For $\mathcal{V}$ a 2-category with all small limits, the 2-stack of all locally constant $\mathcal{V}$-valued stacks on S will be denoted $\mathrm{St}_{\mathcal{V}}(S)^L$.

8.10. Remark. Note that $\mathrm{St}_{\mathcal{V}}(S)^L$ is a 2-stack on S.

8.11. The next proposition gives an equivalent description of locally constant **Gpd**-stacks, which furthermore reveals their structure.

8.12. Proposition. *The 2-category of* **Gpd**-*valued locally constant stacks on S, as defined in 8.8, is equivalent to the 2-category of* **Gpd**-*valued stacks that are locally constant (definition 7.5).*

Proof. See [12], remarks in section 2. □

Part 3. Group actions

9. Group actions in 3-categories

9.1. Definition. Let G be a (discrete) group. Let M be an object of a 2-category $\mathcal{C}$. An *action* of G on M is a functor (between 2-categories):

$$\rho\colon \mathbf{B}G \to \mathcal{C}$$

such that $\rho(*) = M$.

9.2. In other words, an action of G on M is a morphism of pointed 3-categories

$$(\mathbf{B}G, *) \to (\mathcal{C}, M).$$

9.3. Definition. The 3-category of *objects of* $\mathcal{C}$ *with an action by* G is the category of functors $[\mathbf{B}G, \mathcal{C}]$.

9.4. Thus given a 2-groupoid M with an action by G, that is, a functor

$$\rho\colon \mathbf{B}G \to 2\mathbf{Gpd}$$

we obtain, for each $g \in G$, a morphism

$$\rho(*) = M \xrightarrow{\rho(g)} M = \rho(*)$$

that we will simply denote g. So just like for actions of groups on sets, we have for each object $m \in M$, and each $g \in G$, an object $g \cdot m = \rho_g(m)$. Given g and h in G, we obtain two automorphisms of M:

$$M \xrightarrow{\quad gh \quad} M$$

$$M \xrightarrow{h} M \xrightarrow{g} M$$

however these are not required to to be equal. Rather, it is required that there exists a natural transformation

$$\rho(g,h)\colon\ \rho(gh) \to \rho(g)\rho(h).$$

At the level of objects, this means that for $m \in M$, the equation

$$g \cdot (h \cdot m) = gh \cdot m$$

does not hold, but that there is a natural equivalence

$$g \cdot (h \cdot m) \to gh \cdot m.$$

allowing us to compare $gh \cdot m$ and $g \cdot (h \cdot m)$. Applying the definition of a weak functor further gives conditions on $\rho(g,h)$. Furthermore, G-equivariant morphisms of 2-groupoids $M \to N$ are required to map these natural equivalences in M to the corresponding ones in N (up to further 2-morphisms in N).

Part 4. Grothendieck construction for 3-groupoids

9.5. Let C be a (3,1)-category. This part will look at functors of type

$$C^{\mathrm{op}} \to 2\mathbf{Gpd}.$$

The main result of this part is that such functors can be reconstructed as a colimit of representable functors. This is a direct generalisation of the Grothendieck construction for 1-categories.

10. Pointed 2-groupoids

10.1. Definition. The 3-category of *pointed* 2-*groupoids* is defined as the coslice (or "under-category") of $2\mathbf{Gpd}$ under $\mathbf{1}$ (the trivial 2-groupoid); it is denoted $(2\mathbf{Gpd})_\bullet$.

10.2. More explicitly:

{0} The objects of $(2\mathbf{Gpd})_\bullet$ are pairs (M^0, m^0) where M^0 is a 2-groupoid and m is a morphism

$$m^0 \colon \mathbf{1} \to M^0.$$

{1} The morphisms $(M^0_1, m^0_1) \to (M^0_2, m^0_2)$ are pairs (M^1, m^1) where M^1 is a morphism

$$M^1 \colon M^0_1 \to M^0_2$$

and m^1 is an equivalence

$$m^1 \colon (M^0_1 \circ m^0_1) \to m^0_2.$$

{2} The 2-morphisms $(M^1_1, m^1_1) \to (M^1_2, m^1_2)$ are pairs (M^2, m^2) with

$$M^2 \colon M^1_1 \to M^1_2$$

and m^2 is an equivalence (abusing notation):

$$m^2 \colon (m^1_2 \circ M^2) \to m^1_1.$$

{3} The 3-morphisms $(M_1^2, m_1^2) \to (M_2^2, m_2^2)$ are

$$M^3 \colon M_1^2 \to M_2^2$$

such that

$$m_2^2 \circ M^3 = m_1^2.$$

11. Grothendieck construction category

11.1. Definition. Let P be a functor $C^{\text{op}} \to 2\mathbf{Gpd}$. Define the *Grothendieck construction of P* as the 3-category that is the opposite of the limit of the following diagram:

$$\begin{array}{ccc} & & (2\mathbf{Gpd})_\bullet \\ & & \downarrow \\ C^{\text{op}} & \xrightarrow[P]{} & 2\mathbf{Gpd} \end{array}$$

where $(2\mathbf{Gpd})_\bullet \to 2\mathbf{Gpd}$ is the natural projection. The Grothendieck construction of P is denoted $\int P$.

11.2. By definition, $\int P$ is equipped with a natural projection

$$\pi_P \colon \int P \to C.$$

11.3. Proposition. *Any functor $C^{\text{op}} \to 2\mathbf{Gpd}$ is a colimit of representable functors $C^{\text{op}} \to 2\mathbf{Gpd}$.*

11.4. First we give the diagram over which a functor $P \colon C^{\text{op}} \to 2\mathbf{Gpd}$ will be a colimit. Then, we construct a colimiting cone over that diagram, which will simultaneously give an explicit description of the objects and morphisms of $\int P$, while proving proposition 11.3. Note that the explicit description of $\int P$ is computed by applying the universal property of the limit defining $\int P$ to maps

$$R \to \int P$$

where R is in turn

- $w_0 = \mathbf{1}$, the trivial 2-groupoid,
- w_2, the walking morphism,
- w_3, the walking 2-morphism, etc.

11.5. The colimit diagram. Let Y denote the Yoneda embedding

$$Y\colon C \to [C^{\mathrm{op}}, 2\mathbf{Gpd}].$$

We shall show that π_P is a diagram in C that represents P. Let $\mathcal{D}$ denote the composite functor $Y \circ \pi_P$:

$$\int P \xrightarrow{\pi_P} C \xrightarrow{Y} [C^{\mathrm{op}}, 2\mathbf{Gpd}].$$

Then we shall show:

$$\varinjlim \mathcal{D} \simeq P.$$

by naturally constructing a cone

$$\lambda\colon \mathcal{D} \to P.$$

11.6. Objects of $\int P$. An object of $\int P$ (the data specifying it) is equivalent to a pair (C^0, T^0) where C^0 is an object of $\mathcal{C}$ and T^0 is a morphism

$$\mathbf{1} \to P(C^0).$$

Which is equivalent to specifying an object T^0 of $P(C^0)$. Given this object, the image $\mathcal{D}(C^0, T^0)$ under $\mathcal{D}$ is the representable functor associated to C^0: $Y(C^0)$. By the Yoneda lemma, a cone component at $\mathcal{D}(C^0, T^0)$:

$$Y(C^0) \to P$$

is equivalent to specifying an object of $P(C^0)$, which is naturally chosen to be T^0. That is, we set $\lambda(C^0, T^0)$ to be the image of T^0 under the Yoneda equivalence.

11.7. Morphisms of $\int P$. The morphisms $(C^0_1, T^0_1) \to (C^0_2, T^0_2)$ are specified by pairs (C^1, T^1) where

$$C^1\colon C^0_2 \to C^0_1,$$

and

$$T^1 \colon P(C^1) \circ T_1^0 \to T_2^0.$$

This is equivalent to specifying a morphism in $P(C_2^0)$:

$$T^1 \colon P(C^1)(T_1^0) \to T_2^0$$

for the objects

$$T_1^0 \in P(C_1^0) \quad T_2^0 \in P(C_2^0).$$

Under the Yoneda equivalence, this provides the data for the cone-component

$$\lambda(C^1, T^1) \colon \lambda(C_2^0, T_2^0) \circ \mathcal{D}(C^1, T^1) \to \lambda(C_1^0, T_1^0).$$

11.8. 2-Morphisms of $\int P$. The 2-morphisms $(C_1^1, T_1^1) \to (C_2^1, T_2^1)$ are pairs (C^2, T^2) where C^2 is a 2-morphism

$$C_2^1 \to C_1^1$$

and T^2 is (suppressing lower morphisms):

$$T^2 \colon P(C^2) \circ T_1^1 \to T_2^1.$$

This is equivalent to specifying a 2-morphism in $P(C_2^0)$:

$$T^2 \colon T_2^1 \circ P(C^2)(T_1^0) \to T_1^1.$$

Under the Yoneda lemma equivalence, this provides data for the cone-component $\lambda(C^2, T^2)$.

11.9. 3-Morphisms of $\int P$. The 3-morphisms $(C_1^2, T_1^2) \to (C_2^2, T_2^2)$ are specified by pairs (C^3, T^3) where C^3 is a morphism

$$C_2^2 \to C_1^2$$

and T^3 is, suppressing lower morphisms,

$$T^3 \colon P(C^3) \circ T_1^2 \to T_2^2.$$

This 3-morphism provides the data, under the Yoneda equivalence for the cone-component $\lambda(C^3, T^2_1)$.

11.10. The properties required for λ to be a universal cone are derived from the composition properties in $\int P$. We give an example calculation. The other ones are very similar.

Consider two composable 1-morphisms of $\int P$:

$$(C^1_1, T^1_1) \quad (C^1_2, T^1_2)$$

with composite:

$$(C^1_{12}, T^1_{12})$$

between the objects:

$$(C^0_1, T^0_1) \to (C^0_2, T^0_2) \to (C^0_3, T^0_3).$$

Then because P is a functor there is an equivalence

$$a\colon P(C^1_2)(P(C^1_1)(T^0_1)) \to P(C^1_2 \circ C^1_1)(T^0_1).$$

By definition of composition in $\int P$, there is an equivalence

$$T^1_{12} \circ a \to T^1_2 \circ P(C^1_2)(T^1_1)$$

which, under the Yoneda lemma equivalence, provides the equivalence

$$\lambda(C^1_1, T^1_1) \circ \lambda(C^1_2, T^1_2) \to \lambda(C^1_{12}, T^1_{12}).$$

Part 5. The fundamental 2-groupoid of a scheme

In this part we define $\Pi_2(X)$ for X a scheme and then prove some fundamental properties about this pro-finite-2-groupoid.

12. Shape of a scheme

12.1. Context. For the whole of this section, we will consider:

- a field k;
- a scheme X over the field k;
- the étale site $X_{\text{ét}}$ of X.

12.2. Shape functor of $X_{\text{ét}}$. Consider the functor

$$s_X : 2\mathbf{Gpd} \xrightarrow{\text{const}_{X_{\text{ét}}}} \mathrm{St}_{2\mathbf{Gpd}}(X_{\text{ét}}) \xrightarrow{\Gamma} 2\mathbf{Gpd}$$

that is the composite of

- const_X: the functor that associates to any 2-groupoid M the constant stack on $X_{\text{ét}}$ with stalk M. This is the 2-stackification of the constant pre-2-stack with constant value M.
- Γ: the global sections functor, which associates to any object $\mathcal{F}$ of $\mathrm{St}_{2\mathbf{Gpd}}(X_{\text{ét}})$ the 2-groupoid $\mathcal{F}(X)$.

The functor s_X is called the *shape* functor of X.

12.3. Proposition. *The shape functor s_X preserves finite limits.*

Proof. Indeed Γ (being a geometric morphism of $(3,1)$-topoi) preserves all limits, and const_X is a left-exact functor, being the left-adjoint component of Γ. □

12.4. Basic idea. The basic idea is the following. The shape functor s_X is a functor of type

$$2\mathbf{Gpd} \to 2\mathbf{Gpd},$$

and at a 2-groupoid T it is computed by $\Gamma(\text{const}_{X_{\text{ét}}}(T))$, which by 8.12 is the 2-groupoid of locally constant T-valued stacks on $X_{\text{ét}}$.

Thus if it were possible to associate to the scheme X some 2-groupoid M_X that co-represented locally constant stacks on X, then that M_X would co-represent s_X as a functor $2\mathbf{Gpd} \to 2\mathbf{Gpd}$.

This isn't the case, but by the Grothendieck construction for 2-groupoids, we can obtain a sort of approximation of this: the shape functor s_X is a colimit of representables. The fundamental 2-groupoid of X is then the pro-2-groupoid that co-represents s_X. The only concern is size issues, which we resolve by restricting the diagram $\mathcal{D}$ to 2-groupoids of a certain size.

12.5. Definition. Apply the Grothendieck construction to the functor

$$s_X\colon (2\mathbf{Gpd}^{\mathrm{op}})^{\mathrm{op}} \to 2\mathbf{Gpd}$$

The resulting Grothendieck construction has a natural projection

$$\pi_{s_X}\colon \int s_X \to 2\mathbf{Gpd}^{\mathrm{op}}$$

Let A be the sub-category of $\int s_X$ consisting of objects T such that $\pi_{s_X}(T)$ is a finite 2-groupoid (finite number of 2-morphisms). Then denote the composite

$$A^{\mathrm{op}} \rightarrowtail (\textstyle\int s_X)^{\mathrm{op}} \xrightarrow{\pi_{s_X}} 2\mathbf{Gpd}$$

by $\Pi_2^{\text{ét}}(X)$. This is the definition of the *étale 2-groupoid of X*.

12.6. Proposition. *For all schemes X, $\Pi_2(X)$ is a pro-finite-2-groupoid.*

Proof. To prove this claim we need to show that A is cofiltered. That is, we need to show that any finite diagram in A has a cone.

Let $D\colon F \to \int s_X$ be a finite diagram in $\int s_X$. We will use the following notation: for an i-morphism f of F, denote by (C_f^i, T_f^i) the object $D(f)$ of $\int s_X$.

Composing D with π_{s_X} (the natural projection to $2\mathbf{Gpd}^{\mathrm{op}}$) gives a finite diagram in $2\mathbf{Gpd}^{\mathrm{op}}$ whose cocones are the cones of the corresponding opposite diagram E in $2\mathbf{Gpd}$. Let $L := \varprojlim(E)$ be the limit of this diagram, thus also a cone. Composing with s_X gives us a cone in $2\mathbf{Gpd}$ which is also a limit, since s_X preserves finite limits.

For any i-morphisms f in F, the system $(T_f^i \mid 0 \leqslant i \leqslant 3)$ produces a cone on E with vertex $\mathbf{1}$, and hence, by L being the limit of E, an object of $s_X(L)$, thus defining an object of $\int s_X$. By construction this is a cocone for D in $\int s_X$. This proves that $\int s_X$ is cofiltered.

Note that the limit L of E is a finite 2-groupoid whenever the diagram E is only composed of finite 2-groupoids, and thus A is cofiltered. □

12.7. Essential property of $\Pi_2(X)$. Recall that the essential property of the pro-finite étale fundamental group of Grothendieck is that it classifies locally constant finite sheaves on X. Classically, this is stated as a correspondence between the finite étale covers of X and *continuous* actions of $\pi_1^{\text{ét}}(X,x)$ on a finite sets. If we consider a pro-finite group to be a pro(finite group), that is, a pro-object in the category of finite groups, then the category of finite sets equipped with a continuous action by $\pi_1^{\text{ét}}(X,x)$ is equivalent to the category of morphisms of pro-objects

$$\mathbf{B}\pi_1^{\text{ét}}(X,x) \to \mathbf{FinSet},$$

where **FinSet** is considered to be a constant pro-object. A similar property holds more generally for $\Pi_2(X)$.

12.8. Theorem. *For any finite 2-groupoid $\mathcal{C}$, there is an equivalence of 3-categories between the category of locally constant $\mathcal{C}$-valued stacks on $X_{ét}$ and the 3-category of morphisms of pro-2-groupoids*

$$\Pi_2(X) \to \mathcal{C}.$$

This equivalence is natural in $\mathcal{C}$.

Here $\mathcal{C}$ is considered to be a constant pro-2-groupoid.

Proof. By Proposition 8.12 and the definition of s_X, $s_X(\mathcal{C})$ is equivalent to the category of locally constant $\mathcal{C}$-valued stacks on $X_{\text{ét}}$.

By Proposition 11.3, $s_X(\mathcal{C})$ is equivalent to

$$\varinjlim\left(\int s_X \xrightarrow{\pi_{s_X}} \mathbf{2Gpd}^{\text{op}} \xrightarrow{Y} \mathbf{2Gpd}^{\mathbf{2Gpd}}. \right)$$

However for any object (M,T) of $\int s_X$, that is, any 2-groupoid M and any locally constant stack with values in M: $T \in s_X(M)$, the corresponding representable functor $2\mathbf{Gpd} \to 2\mathbf{Gpd}$ is $2\mathbf{Gpd}(M,-)$. The value at $\mathcal{C}$ is this functor is category of functors

$$M \to \mathcal{C}.$$

Since $\mathcal{C}$ is a finite 2-groupoid, there is a finite groupoid M_0 and an epi $M \to M_0$ such that any functor $M \to \mathcal{C}$ factors as

$$M \to M_0 \to \mathcal{C}.$$

Thus, pre-composing with the inclusion $A \rightarrowtail (\int s_X)$ will not alter the colimit. Notice that this is the definition of the hom-3-category of morphisms of pro-2-groupoids

$$\Pi_2(X) \to \mathcal{C}$$

because, since $\mathcal{C}$ is constant, this amounts to the colimit (over the opposite of the diagram giving $\Pi_2(X)$) of the hom-3-categories between the components of $\Pi_2(X)$ and $\mathcal{C}$. □

12.9. Notation. Given a locally constant $\mathcal{C}$-valued stack $\mathcal{F}$ on $X_{\text{ét}}$, the corresponding functor $\Pi_2(X) \to \mathcal{C}$ will be denoted $\Pi_2(\mathcal{F})$.

12.10. Proposition. *For a scheme X, the equivalence class of the pro-finite-2-groupoid* $\Pi_2(X)$ *is uniquely determined by the property it satisfies in Theorem 12.8.*

Proof. Indeed $\Pi_2(X)$ is a pro-finite-2-groupoid that co-represents s_X. If T was another pro-finite 2-groupoid with the same property, then for any finite 2-groupoid $\mathcal{C}$, there would be an equivalence that is natural in $\mathcal{C}$:

$$[T,\mathcal{C}] \to [\Pi_2(X),\mathcal{C}].$$

Therefore, by lemma 5.4, there is an equivalence of pro-finite-2-groupoids

$$T \to \Pi_2(X).$$

□

12.11. Proposition. *The étale fundamental-2-groupoid construction is a functor from the category of schemes to the category of pro-2-groupoids.*

Proof. The hard part was the existence of the fundamental 2-groupoid. Since $\Pi_2(X)$ is defined, for X a scheme, by a universal property, the functoriality is relatively straightforward. Indeed given any morphism of schemes $X \to Y$, the inverse image 2-stack functor takes the constant 2-stack $\mathrm{const}_Y(\mathcal{C})$ to $\mathrm{const}_X(\mathcal{C})$, and therefore, the inverse-image for stacks takes locally constant $\mathcal{C}$-valued stacks on Y to locally constant $\mathcal{C}$-valued stacks on X:

$$f_{\mathcal{C}}^{-1}\mathrm{St}_{\mathcal{C}}(Y_{\text{ét}})^L \to \mathrm{St}_{\mathcal{C}}(X_{\text{ét}})^L$$

Because of const_X is a functor, $f_{\mathcal{C}}^{-1}$ is functorial in $\mathcal{C}$, that is, is a natural transformation. Thus we have a natural transformation of representable functors

$$[\Pi_3(Y), -] \to [\Pi_2(X), -]$$

of type **Fin**2**Gpd** $\to$ 2**Gpd** that are co-represented by $\Pi_2(X)$ and $\Pi_2(Y)$ respectively. Thus, by lemma 5.4, this corresponds to a morphism of pro-finite-2-groupoids:

$$\Pi_2(X) \to \Pi_2(Y).$$

□

12.12. Remark. By the above proof we also deduce that given a morphism of schemes $f\colon X \to Y$, the resulting morphism of pro-2-groupoids $\Pi_2(f)$ has the following property: given a locally constant $\mathcal{C}$-valued stack $\mathcal{F}$ on $Y_{\text{ét}}$, represented by:

$$\Pi_2(\mathcal{F})\colon \Pi_2(Y) \to \mathcal{C}$$

then the composite

$$\Pi_2(X) \xrightarrow{\Pi_2(f)} \Pi_2(Y) \xrightarrow{\Pi_2(\mathcal{F})} \mathcal{C}$$

represents the locally constant stack $f^{-1}\mathcal{F}$ on X.

12.13. Lemma. *The étale fundamental groupoid of a scheme X is uniquely determined by the property that the category of locally constant finite stacks in groupoids is equivalent to the category of functors*

$$\Pi_2(X) \to \mathbf{FinGpd}$$

Proof. A reformulation of Cayley's theorem for finite groups says that for any finite groupoid M, M is equivalent to a sub-groupoid of the core of finite sets:

$$M \rightarrowtail \mathrm{core}(\mathbf{FinSet}).$$

(The core of a category is the groupoid obtained by removing all non-invertible morphisms.)

A generalisation is true for 2-groupoids: for any finite 2-groupoid M, M is equivalent to a sub-2-groupoid of the core of the 2-category of finite groupoids. We construct the functor $M \to \mathbf{FinGpd}$ that gives such an equivalence. First, consider the case when M is connected. We may therefore assume that M has a single object m_0. Then

$$[M(-,m_0), M(-,m_0)] \simeq M(m_0,m_0)$$

by the Yoneda lemma. Furthermore since M has a single object, $[M(-,m_0), M(-,m_0)]$ is a sub-2-groupoid of $[M(m_0,m_0), M(m_0,m_0)]$. In general any finite 2-groupoid M is a finite co-product $F_1 + \cdots + F_n$ of connected finite 2-groupoids F_i. Each F_i is equivalent to a sub-2-groupoid of **FinGpd**. Choosing distinct representatives of equivalence classes of 1-groupoids if necessary, one can make sure the morphisms $F_i \rightarrowtail \mathbf{FinGpd}$ do not overlap. In this way we can obtain an equivalence of M and a sub-2-groupoid of **FinGpd**.

Furthermore, **FinGpd** is the colimit of the diagram of inclusions of all its finite sub-2-groupoids.

So pick a representative M_i, $i \in I$, of each equivalence class of finite 2-groupoids. Then for each M_i, pick a sub-2-groupoid of **FinGpd** that it is equivalent to:

$$s(M_i) \rightarrowtail \mathbf{Fin2Gpd}.$$

Then it is enough to apply 12.10 to these $s(M_i)$. □

13. Fields

13.1. Defining the absolute Galois group G_k of a field was in essence the first definition of $\pi_2^{\text{ét}}(X)$ for the special case where X is the spectrum of a field. We get a (connected) pro-finite groupoid by taking the delooping of $\mathbf{B}G_k$. Fields are actually "étale homotopy 1-types", in that they don't actually contain any higher-homotopical data. In the following we will show this at least at level 2, that is, we will show that $\mathbf{B}G_k \simeq \Pi_2(Spec(k))$.

13.2. Let G be a pro-finite group. Recall that $[\mathbf{B}G, \mathbf{Set}]$ is equivalent to the category of sets with a continuous G-action. This category can be equipped with the structure of a site by considering coverings $(U_i \to U \mid i \in I)$ to be families of G-finite sets such that $\varinjlim_{i\in I} U_i \to U$ is surjective.

13.3. Lemma. *Let $G = (G(i) \mid i \in I)$ be a pro-finite group with surjective transition maps, indexed by some category I. Let $\mathcal{C}$ be a finite 2-groupoid. Then there is an equivalence of categories between the category of locally constant $\mathcal{C}$-stacks on $[\mathbf{B}G, \mathbf{Set}]$ and $[\mathbf{B}G, \mathcal{C}]$.*

Proof. Let i be an object of I. Consider the Yoneda embedding

$$Y\colon \mathbf{B}G(i) \rightarrowtail [\mathbf{B}G(i)^{\mathrm{op}}, \mathbf{Set}],$$

and let $m(i)$ denote $Y(*_i)$ where $*_i$ is the unique object of $\mathbf{B}G(i)$. By the Yoneda lemma, each $m(i)$ is a $G(i)$-set such that its group of automorphisms is isomorphic to $G(i)$. Now each $m(i)$ defines an object of $[\mathbf{B}G, \mathbf{Set}]$, and so the system $(m(i) \mid i \in I)$ defines a pro-object M in $[\mathbf{B}G, \mathbf{Set}]$, which has an automorphism pro-group isomorphic to G.

Any $\mathcal{C}$-valued stack on $[\mathbf{B}G, \mathbf{Set}]$ thus produces, by taking its value at M, a pro-object in $\mathcal{C}$ with an action by G.

Let $\mathcal{S}$ be a locally constant $\mathcal{C}$-valued stack on $[\mathbf{B}G, \mathbf{Set}]$:

$$\mathcal{S}\colon [\mathbf{B}G, \mathbf{Set}]^{\mathrm{op}} \to \mathbf{FinGpd}.$$

Then because $\mathcal{S}$ is locally constant, there is a cover U_j such that $\mathcal{S}$ is constant over each U_j. We may without loss of generality assume that each U_j is a connected G-set. Each U_j is therefore covered by $m(i(j))$ for some $i(j) \in I$. Since I is cofiltered, $\mathcal{S}$ is therefore constant over some $m(i_0)$, $i_0 \in I$. We obtain in this fashion an object of $\mathcal{C}$: $\mathcal{S}(m(i_0))$ with an action by $G(i_0)$ (since $m(i_0)$ has automorphism group $G(i_0)$), and hence an action by G. This is equivalent to an object of $[\mathbf{B}G, \mathcal{C}]$.

Conversely given an object M of $\mathcal{C}$ with an action by G, this action must factor through some component of G:

$$\mathbf{B}G \to \mathbf{B}G(i_0) \to \mathcal{C}$$

for some $i_0 \in I$.

We can then construct a locally constant $\mathcal{C}$-valued stack on $[\mathbf{B}G, \mathbf{Set}]$ using the covering

$$m(i_0) \to \mathbf{1}$$

of $m(i_0)$ of the terminal object, by giving the value at $m(i_0)$, which is M, and descent data. In this case the descent data is equivalent to the action of $G(i_0)$ on M. □

13.4. Proposition. *For any field k, $\Pi_2(\mathrm{Spec}(k)) \simeq \mathbf{B}G_k$, where G_k is the absolute Galois group of k.*

Proof. Recall that the étale site of a scheme is equivalent to the canonical site on the category of G_k-sets:

$$\mathrm{Spec}(k)_{\text{ét}} \simeq [\mathbf{B}G_k, \mathbf{Set}].$$

Therefore, for some finite 2-groupoid $\mathcal{C}$, the category of locally constant $\mathcal{C}$-stacks on $\mathrm{Spec}(k)_{\text{ét}}$ is equivalent to $[\mathbf{B}G_k, \mathcal{C}]$. Thus by 12.10

$$\Pi_2(\mathrm{Spec}(k)) \simeq \mathbf{B}G_k.$$

□

13.5. Remark. For any scheme X over a field k, there is a structure map $X \to \mathrm{Spec}(k)$ and thus, by functoriality of Π_2, a morphism of pro-2-groupoids

$$\Pi_2(X) \to \Pi_2(\mathrm{Spec}(k)) \simeq \mathbf{B}G_k.$$

14. Relative schemes

14.1. Lemma. *Let X be a scheme over a field k. Let $K|k$ be a finite Galois extension of k, and let $X_K := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. Then there is an action of $G_{K|k}$ on $\Pi_2(X_K)$.*

Proof. For any $g \in G_{K|k}$, there is a corresponding morphism of schemes

$$\mathrm{Spec}(K) \to \mathrm{Spec}(K)$$

that is over $\mathrm{Spec}(k)$. This lifts to the fibre product to a morphism of schemes

$$f_g \colon X_K \to X_K$$

over X_K, and of course this is functorial, that is, this provides an action of G_K on the object $X_K \to X$ in the category of schemes over X. Since Π_2 is a functor, the images $\Pi_2(f_g)$ of these morphisms give an action of $G_{K|k}$ on $\Pi_2(X_K)$ (again over $\Pi_2(X)$). □

14.2. Lemma. *For any scheme X over a field k, and any finite Galois extension $K|k$, the category of locally constant $\mathcal{C}$-valued stacks on X is equivalent to the category of $G_{K|k}$-morphisms*

$$\Pi_2(X_K) \to \mathcal{C}.$$

In this statement, $\mathcal{C}$ is equipped with the trivial $G_{K|k}$-action.

Proof. Let $X_K := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$. The morphism

$$X_K \to X$$

is étale, and can be used to compute, by using descent in the 2-stack of locally constant $\mathcal{C}$-valued stacks, the category of locally constant stacks on X by Galois descent of locally constant stacks on X_K. Indeed the pre-2-stack of locally constant stacks on X is a stack, and applying the stack-condition on the cover $X_K \to X$ produces the familiar descent category: locally constant stacks on X can be described by the data of a locally constant stack $\mathcal{F}$ on X_K, and a collection of equivalences

$$(\alpha_g : g^{-1}\mathcal{F} \to \mathcal{F} \mid g \in G_{K|k})$$

etc. In this case, the morphisms $g^{-1}\mathcal{F} \to \mathcal{F}$ correspond to

$$\Pi_2(\mathcal{F}) \circ \Pi_2(f_g) \to \Pi_2(\mathcal{F}),$$

by 12.12. Thus this descent category is equivalent to the category of $G_{K|k}$-equivariant morphisms $\Pi_2(X_K) \to \mathcal{C}$, for the action of $G_{K|k}$ on $\Pi_2(X_K)$. □

14.3. Lemma. *Let $K|k$ be a finite Galois extension. Then the following is a fibred product:*

$$\begin{array}{ccccc} & & \mathbf{B}G_K & & \\ & \swarrow & & \searrow & \\ \mathbf{B}G_k & & & & \mathbf{1} \\ & \searrow & & \swarrow & \\ & & \mathbf{B}G_{K|k}. & & \end{array}$$

Proof. In the above diagram, $\mathbf{B}G_K$ being equivalent to the homotopy fibre of $\mathbf{B}G_k \to \mathbf{B}G_{K|k}$ is equivalent to G_K being the kernel of $G_k \to G_{K|k}$. □

14.4. Lemma. *Let X be a scheme over k and let $K|k$ be a finite Galois extension. Then the following is a fibred product:*

$$\begin{array}{ccc} \Pi_2(X_K) & \longrightarrow & \mathbf{B}G_K \\ \downarrow & & \downarrow \\ \Pi_2(X) & \longrightarrow & \mathbf{B}G_k \end{array}$$

Proof. Two out of the three rectangles are fibred products in the following diagram:

$$\begin{array}{ccccc} \Pi_2(X_K) & \longrightarrow & \mathbf{B}G_K & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow & & \downarrow \\ \Pi_2(X) & \longrightarrow & \mathbf{B}G_k & \longrightarrow & \mathbf{B}G_{K|k} \end{array}$$

and so

$$\begin{array}{ccc} \Pi_2(X_K) & \longrightarrow & \mathbf{B}G_K \\ \downarrow & & \downarrow \\ \Pi_2(X) & \longrightarrow & \mathbf{B}G_k \end{array}$$

is also a fibred product. □

15. Gerbes

15.1. In this section we study how $\Pi_2(X)$ classifies A-gerbes, for some sheaf of groups A on $\mathrm{Spec}(k)$. The basic reference for the theory of gerbes is [8]. A more modern treatment is given in [7] and [6].

15.2. Notation. If M is a groupoid, $\mathrm{Aut}(M)$ denotes the sub-2-category of **Gpd** composed of the object M and the auto-equivalences of M.

15.3. Lemma. *Let X be a scheme over a field k. Let A be a finite abelian group (no action by G_k). Then the category of A-gerbes on X is equivalent to the category of functors*

$$\Pi_2(X) \to \mathrm{Aut}(\mathbf{B}^1 A).$$

Proof. In this case A is constant over X. The 2-category of A-gerbes on X is the 2-category of stacks that are locally equivalent to the stack of A-torsors. The stack of A-torsors is equivalent to $\mathrm{const}(\mathbf{B}A)$. Thus choosing a cover over which the A-gerbe is trivial, we get a "cocycle with values in $\mathrm{Aut}(\mathrm{const}(\mathbf{B}A)) \simeq \mathrm{const}(\mathrm{Aut}(\mathbf{B}A))$. That is, a locally constant stack with values in $\mathrm{Aut}(\mathbf{B}A)$. We make this more precise.

Let P be an A-gerbe. Following Breen in [6], we present P using the so-called "semi-local description" (see section 4 of [6]) over a cover $(U_i \mid i \in I)$ such that over each U_i we have an equivalence

$$\varphi_i \colon P|_{U_i} \simeq \mathrm{Tors}(A)|_{U_i}.$$

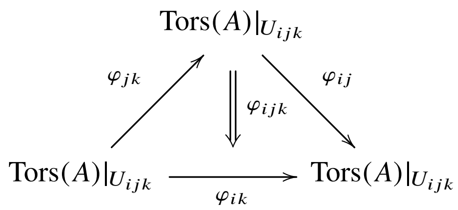
Choosing quasi-inverses of the φ_i, we get

- an induced family of equivalences

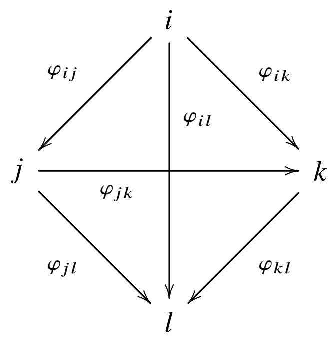
$$\varphi_{ij} := \varphi_i|_{U_{ij}} \circ \varphi_j^{-1}|_{U_{ij}} \colon \mathrm{Tors}(A)|_{U_{ij}} \to P|_{U_{ij}} \to \mathrm{Tors}(A)|_{U_{ij}}$$

  above each U_{ij}.

- Natural transformations φ_{ijk} over each U_{ijk}:



- That satisfy the following coherence condition (commuting diagram):



  (where each of the vertices i, j, k, l is $\mathrm{Tors}(A)|_{U_{ijkl}}$, and each face with vertices $\{x, y, z\} \subset \{i, j, k, l\}$ is the 2-morphism φ_{xyz}).

However the stack of A-torsors is equivalent to the stack $\mathrm{const}_{X_{\text{ét}}}(\mathbf{B}A)$ and for each U, the equivalences

$$\mathrm{const}(\mathbf{B}A)|_U \to \mathrm{const}(\mathbf{B}A)|_U$$

are equivalent to equivalences of $\mathbf{B}A$, that is, morphisms in $\mathrm{Aut}(\mathbf{B}A)$. Thus the presentation given above is simply descent data for a global section of $\mathrm{const}_{X_{\text{ét}}}(\mathrm{Aut}(\mathbf{B}A))$. Thus the A-gerbes are equivalent to locally constant $\mathrm{Aut}(\mathbf{B}A)$-valued stacks on $X_{\text{ét}}$. By the universal property of $\Pi_2(X)$, therefore, they are equivalent to functors $\Pi_2(X) \to \mathrm{Aut}(\mathbf{B}A)$. □

15.4. Lemma. *Let X be a scheme over a field k. Let A be a finite G_k-module. Let $K|k$ be a finite Galois extension such that A is a trivial G_K-module. Then the category of A-gerbes over $X_{ét}$ is equivalent to the $G_{K|k}$-action morphisms*

$$\Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}A).$$

Proof. The sheaf of abelian groups A on $\mathrm{Spec}(k)$ pulls back to a sheaf of abelian groups on X. The A-gerbes on $X_{\text{ét}}$ form a 2-stack, so we may apply the descent condition over the cover $X_K \to X$, that is, A-gerbes over X may be defined via Galois descent of A-gerbes over X_k.

Because the sheaf of abelian groups A is constant over $\mathrm{Spec}(K)_{\text{ét}}$, the same is true for the pull-back to X_K, and so we may apply lemma 15.3, so that the A-gerbes over X_K are equivalent to morphisms

$$\Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}A).$$

Applying Galois descent under the action of the group $G_{K|k}$ implies that the A-gerbes over X are equivalent to the $G_{K|k}$-morphisms

$$\Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}A).$$

□

Part 6. Pseudo-geometric functors

15.5. Context. Consider the following setting for this section:

- k is a field.
- $\bar{k}$ is an arbitrary algebraic closure of k.
- X is a scheme over k.
- $\bar{X} := X \otimes_k \bar{k}$.

16. Chern classes of line bundles

16.1. Line bundles, which are structured sheaves on the étale topology, are not locally constant in any way. Thus $\Pi_2(X)$ (or any higher variant) does not transport all of the underlying data contained within them (see part 9 for a more on this). An approximation is however achieved via the Chern class, which can be transported along morphisms of Π_2.

16.2. Let L be a line bundle on X. The (étale) stack of line bundles on X is equivalent to the stack of $\mathbf{G}_{\mathrm{m}}$-torsors on X, thus we may consider L to be a $\mathbf{G}_{\mathrm{m}}$-torsor. Consider, for some $n \geqslant 1$, the Kummer short exact sequence K_n of sheaves of abelian groups on $X_{\text{ét}}$:

$$\mu_n \longrightarrow \mathbf{G}_{\mathrm{m}} \xrightarrow{\mathrm{m}_n} \mathbf{G}_{\mathrm{m}}$$

Here μ_n is the sheaf of n-th roots of unity, and m_n is the n-th power morphism.

Using this sequence, L produces, for all $n \geqslant 0$, a μ_n-gerbe. This gerbe is represented by a stack on $X_{\text{ét}}$: it is the stack $c_n(L)$ that associates to any étale cover U of X the groupoid $c_n(L)(U)$:

$\{0\}$ objects are pairs (M, a) where

- M is a $\mathbf{G}_{\mathrm{m}}$-torsor on $X_{\text{ét}}$;
- $a\colon M^{\otimes n} \to L$ is an isomorphism of $\mathbf{G}_{\mathrm{m}}$-torsors;

{1} Morphisms $(M_1, a_1) \to (M_2, a_2)$ are morphisms $f\colon M_1 \to M_2$ of $\mathbf{G}_{\mathrm{m}}$-torsors, such that the following diagram commutes:

$$\begin{array}{ccc} M_1^{\otimes n} & \xrightarrow{f^{\otimes n}} & M_2^{\otimes n} \\ & {\scriptstyle a_1}\searrow \quad \swarrow{\scriptstyle a_2} & \\ & L. & \end{array}$$

16.3. Consider the category $\overrightarrow{\mathbf{N}}_{\geqslant 1}$:

{0} objects are integers $\geqslant 1$,

{1} $f\colon a \to b$ exists if and only if b divides a.

For each morphism $n \to m$ in $\overrightarrow{\mathbf{N}}_{\geqslant 1}$, since $m|n$ there is some l such that $n = ml$. Then $\mathrm{m}_m \circ \mathrm{m}_l = \mathrm{m}_n$ and there is morphism of short-exact sequences $K_n \to K_m$:

$$\begin{array}{ccccc} \mu_n & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \xrightarrow{\mathrm{m}_n} & \mathbf{G}_{\mathrm{m}} \\ \downarrow & & \downarrow {\scriptstyle \mathrm{m}_l} & & \downarrow {\scriptstyle \mathrm{id}} \\ \mu_m & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \xrightarrow{\mathrm{m}_m} & \mathbf{G}_{\mathrm{m}}. \end{array}$$

Thus there is a functor from $\overrightarrow{\mathbf{N}}_{\geqslant 1}$ to the category of short exact sequences of sheaves on X. The system $(\mu_n \mid n \in \overrightarrow{\mathbf{N}}_{\geqslant 1})$ is a pro-object in the category of abelian sheaves on $X_{\text{ét}}$, and the system $(c_n(L) \mid n \in \overrightarrow{\mathbf{N}}_{\geqslant 1})$ produced above forms a system of μ_n-gerbes compatible with this pro-object. (One could define the notion of an $\hat{\mathbf{Z}}(1)$-gerbe over $X_{\text{ét}}$, but this is enough for our purposes.)

16.4. Definition. For X a scheme and L a line bundle on X, the system $(c_n(L) \mid n \in \overrightarrow{\mathbf{N}}_{\geqslant 1})$ will be denoted $\hat{c}_X(L)$, and called *Chern class of* L. The n-component of $\hat{c}_X(L)$ will be called the *modulo n Chern class* of L and denoted $c_X(L)_n$.

17. Pseudo-geometric morphisms

17.1. Suppose X and Y are two schemes over k. The étale fundamental 2-groupoids are thus equipped with structure morphisms to $\mathbf{B}G_k$. Let

$$\varphi\colon \Pi_2(X) \to \Pi_2(Y)$$

be a morphism of pro-2-groupoids over $\mathbf{B}G_k$. We would like to discuss a property of φ that imposes some geometric rigidity, which is necessary if φ is to lift to a geometric morphism $X \to Y$. In a section that follows we will show that this property is also sufficient for the special case of morphisms $\mathrm{Spec}(k) \to X$ of Brauer-Severi varieties.

17.2. Suppose first that $\varphi := \Pi_2(f)$ does come from a geometric morphism $f\colon X \to Y$. Let $n \in \mathbf{N}_{\geqslant 1}$. Then the modulo n Chern class of L is represented by some $G_{K|k}$-equivariant morphism

$$c_n\colon \Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}\mu_n)$$

for a suitable extension $K|k$ of k.

Consider the composite $c_n \circ \varphi$, which is a $G_{K|k}$-equivariant morphism

$$\Pi_2(X_K) \xrightarrow{\varphi} \Pi_2(Y_K) \xrightarrow{c_n} \mathrm{Aut}(\mathbf{B}\mu_n)$$

of the type that would represent a modulo n Chern class of a line bundle on X. Because in this case φ does come from a geometric morphism f, we have that $c_n \circ \varphi$ represents the line bundle f^*L.

17.3. Definition. Let X and Y be schemes over k and $\varphi\colon \Pi_2(X) \to \Pi_2(Y)$ a morphism over $\mathbf{B}^1G_k$. We say that φ is *pseudo-geometric* if for each line bundle L on Y, there exists some line bundle L' such that for each $n \in \mathbf{N}_{\geqslant n}$ there is an equivalence

$$c_Y(L)_n \circ \varphi \simeq c_X(L')_n.$$

17.4. Remark. Because of Hilbert's theorem 90, a section of the structure morphism $\Pi_2(X) \to \mathbf{B}G_k$ is pseudo-geometric if and only if it transports Chern classes of line bundles to trivial morphisms

$$\mathbf{B}G_k \to \mathrm{Aut}(\mathbf{B}\mu_n).$$

17.5. Remark. For anabelian varieties X and their étale fundamental groups $\pi_1^{\text{ét}}(X, \bar{x})$, and the special case of morphisms of varieties

$$\mathrm{Spec}(k) \to X,$$

the definition of pseudo-geometric functor reduces to the definition of a "good morphism", as introduced by Mohamed Saidi, see definition 1.4.1 of [4], for which it has been useful in studying variants of the sections conjecture (see also [5]). It is therefore not surprising that this is also a useful notion when considering $\Pi_2(X)$.

Part 7. Conjectures

18. Structure

18.1. Let X be a scheme over a field k. Then the structure morphism $\mathrm{st}\colon X \to \mathrm{Spec}(k)$ produces a morphism of pro-finite-2-groupoids:

$$\Pi_2(\mathrm{st})\colon \Pi_2(X) \to \Pi_2(\mathrm{Spec}(k)) \simeq \mathbf{B}G_k.$$

A rational point s of X, that is, a section of the structure morphism:

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\quad s \quad} & X \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{st}} & \\ & \mathrm{Spec}(k), & \end{array}$$

produces, by functoriality of Π_2, a section of $\Pi_2(\mathrm{st})$:

$$\begin{array}{ccc} \mathbf{B}G_k & \xrightarrow{\quad \Pi_2(s) \quad} & \Pi_2(X) \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \Pi_2(\mathrm{st})} & \\ & \mathbf{B}G_k. & \end{array}$$

18.2. Definition. For a scheme X over a field k, let $\mathrm{Sec}_2^{\text{ét}}(X)$ denote the set of equivalence classes of sections of $\Pi_2(\mathrm{st})$.

19. Conjectures

19.1. There is thus a natural map

$$r_2\colon X(k) \to \mathrm{Sec}_2^{\text{ét}}(X)$$

from the set of rational points of X into $\mathrm{Sec}_2^{\text{ét}}(X)$. The same definition applied to $\Pi_1(X)$, which is equivalent, in the case of connected schemes, to $\mathbf{B}\pi_1^{\text{ét}}(X,x)$, where $\pi_1^{\text{ét}}(X)$ is Grothendieck's étale fundamental group, produces $\mathrm{Sec}_1^{\text{ét}}(X)$. There is a commutative diagram

$$\begin{array}{ccc} X(k) & \xrightarrow{r_2} & \mathrm{Sec}_2^{\text{ét}}(X) \\ & \underset{r_1}{\searrow} & \downarrow \\ & & \mathrm{Sec}_1^{\text{ét}}(X). \end{array}$$

Grothendieck's famous "sections conjecture" postulates that r_1 is in fact a bijection in the case that X is an "Anabelian" variety. Since $\mathrm{Sec}_2^{\text{ét}}(X)$ is more refined, we can ask if there is a class of schemes strictly larger than the Anabelian schemes for which r_2 is a bijection.

19.2. Definition. For X a scheme over k, let $\mathrm{Sec}_2^{\text{ét}}(X)^g$ denote the subset of $\mathrm{Sec}_2^{\text{ét}}(X)$ composed of equivalence classes of sections of $\Pi_2(\mathrm{st})$ that are furthermore pseudo-geometric.

19.3. Because rational points are actual morphisms of schemes, they pull-back line bundles, and so for any rational point s, $\Pi_2(s)$ is pseudo-geometric. Therefore r_2 factors through the inclusion $\mathrm{Sec}_2^{\text{ét}}(X)^g \rightarrowtail \mathrm{Sec}_2^{\text{ét}}(X)$, and so $\mathrm{Sec}_2^{\text{ét}}(X)$ is a more refined approximation of $X(k)$. In the case of hyperbolic curves, this subset has already been studied in the context of the étale fundamental group of X, see for example [4].

19.4. It is clear that in general, the set $\mathrm{Sec}_2^{\text{ét}}(X)^g$ has to be further refined if one is to hope to obtain a bijection

$$X(k) \to \mathrm{Sec}_2^{\text{ét}}(X)^g.$$

Indeed for example the affine scheme $\mathbf{A}_k^1$ has plenty of rational points in $\mathbf{A}_k^1(k)$ but

$$\Pi_2(\mathbf{A}_k^1) \simeq \mathbf{B}G_k,$$

and so $\mathrm{Sec}_2^{\text{ét}}(\mathbf{A}_k^1)^g$ is singleton. However we can at least obtain a criterion for the existence of a rational point.

19.5. Conjecture. *There is a large class of schemes X such that X has a rational point if and only if $\mathrm{Sec}_2^{\text{ét}}(X)^g$ is non-empty.*

19.6. Of course if one defined higher fundamental n-groupoids of schemes, then the set $\mathrm{Sec}_n^{\text{ét}}(X)$ of sections of the structure morphism $\Pi_n(X) \to \mathbf{B}G_k$ is refined even further, and the class of schemes for which the criterion works could be further enlarged. This would culminate into the fundamental ∞-groupoid of X, giving rise to $\mathrm{Sec}_\infty^{\text{ét}}(X)$.

19.7. In the following section cronjecture 19.5 is proved for the class of Severi-Brauer varieties.

Part 8. Étale homotopy sections of Severi-Brauer varieties

20. Severi-Brauer varieties

20.1. A Severi–Brauer variety over a field k is an algebraic variety X which becomes isomorphic to a projective space over an algebraic closure of k.

In dimension one, the Severi–Brauer varieties are conics. The conic with equation

$$z^2 = ax^2 + by^2$$

splits, that is, is isomorphic to the projective line over k, if and only if it has a rational point (a k-point).

Since Severi-Brauer varieties become isomorphic to projective space after an extension of scalars, they can be defined and studied purely in terms of Galois descent.

The following material is all classical and can be found in [14].

20.2. Definitions. A *Severi-Brauer variety* of degree $n \geqslant 1$ over a field k is a variety P such that there exists an isomorphism

$$P_{\bar{k}} \simeq \mathbf{P}^{n-1}_{\bar{k}}$$

where $\bar{k}$ is some algebraic closure of k.

For example a Severi-Brauer variety of degree n for a finite field extension $K|k$ is a variety P over k such that $P \otimes_k K \simeq \mathbf{P}^{n-1}$. In this case the variety P is said to be *split* by K. Of course, $\mathbf{P}^{n-1}$ is itself a Severi-Brauer variety of degree n, and it is said to be *trivial*.

20.3. Definition. Let X be a variety over a field k and let K be a Galois extension of k. A line bundle on $X_K := X \otimes_k K$ is *rational* if it descends to a line bundle on X.

20.4. When $X = \mathbf{P}^{n-1}$, we have $\mathrm{Pic} X \simeq \mathbf{Z}$, and this is generated by $\mathcal{O}_{\mathbf{P}_{n-1}}(1)$. Therefore extension of scalars from k to K is an isomorphism

$$\mathrm{Pic}(X) \simeq \mathrm{Pic}(X_K).$$

Thus in this case *every* line bundle on X_K is rational. In the case that X is a non-trivial Severi-Brauer variety, this does not hold, and this produces a criterion for X being trivial or not.

20.5. Notation. Let X be a Severi-Brauer variety over k, and let $K|k$ be a finite field extension which splits X. Denote by φ_K an isomorphism

$$X_K \to \mathbf{P}_K^{n-1}.$$

Furthermore let $\mathcal{O}_{X_K}(1)$ denote

$$\varphi^*(\mathcal{O}_{\mathbf{P}_K^{n-1}}(1)).$$

20.6. Proposition. *Let X be a Severi-Brauer variety of degree n that splits over a finite field extension $K|k$. The line bundle $\mathcal{O}_{X_K}(1)$ is rational if and only if X is trivial.*

Proof. See [23] Proposition 9.1. □

The next proposition relates the triviality of a Severi-Brauer variety with the existence of rational points on that variety.

20.7. Proposition. *A Severi-Brauer variety X over k is trivial if and only if $X(k) \neq \emptyset$.*

Proof. See [23] Proposition 9.7. □

The main difficulty in the sections conjecture is producing a rational point from a section. This is why the result above is central to the method presented below, since it reduces finding a rational point to proving that a certain line bundle is rational (by showing that a certain line bundle descends).

21. Étale sections of Brauer-Severi varieties

21.1. Theorem. *Let X be a Brauer-Severi variety over a characteristic 0 field k. Let s be a (weak) section of the structure morphism $\Pi_2(X) \to \mathbf{B}G_k$ that is furthermore pseudo-geometric. Then X has a rational point.*

Proof.

21.2. Let L be a rational line bundle on X. Then there exists some $n \geqslant 1$ such that

$$\bar{L} \simeq O_{\bar{X}}(n).$$

Consider the modulo n Chern class associated to L. This is a μ_n-gerbe on $X_{\text{ét}}$.

21.3. Let $K|k$ be a Galois extension that contains the n-th roots of unity and furthermore for which we already have the isomorphism:

$$L_K \simeq O_{X_K}(n).$$

Then the modulo n Chern class of L is a μ_n-gerbe on X for the sheaf of groups μ_n, which is constant on X_K, and thus, by Lemma 15.4, this corresponds to a morphism of $G_{K|k}$-groupoids

$$c\colon \Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}\mu_n).$$

21.4. Pulling back by $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ gives the restriction of the μ_n-gerbe c to X_K, and this is represented by a morphism

$$c_K\colon \Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}\mu_n)$$

that is over c, in the sense that the forgetful functor

$$F\colon [\Pi_2(X_K), \mathrm{Aut}(\mathbf{B}\mu_n)]^{G_{K|k}} \to [\Pi_2(X_K), \mathrm{Aut}(\mathbf{B}\mu_n)]$$

that sends $G_{K|k}$-equivariant morphisms to the underlying morphisms sends c to c_K.

21.5. The μ_n-gerbe associated to c has a section over the étale cover $X_K \to X$ given by the pair $(O_{X_K}(1), O_{X_K}^{\otimes n} \simeq L_K)$. Thus over X_K it is equivalent to the trivial μ_n-gerbe (which is the stack of μ_n-torsors represented by the class trivial class $H^2_{\text{ét}}(X_K, \mu_n)$).

The trivial μ_n-gerbe on X_K is represented by the constant (or trivial) morphism

$$\Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}\mu_n)$$

that is, the unique up to equivalence morphism that factors as

$$\mathbf{0}_{X_K} : \Pi_2(X_K) \to \mathbf{1} \to \mathrm{Aut}(\mathbf{B}\mu_n).$$

There is therefore a homotopy between c_K and $\mathbf{0}_K$:

$$a \colon \mathbf{0}_{X_K} \to c_K.$$

However this homotopy a is (a priori) only a homotopy in $[\Pi_2(X_K), \mathrm{Aut}(\mathbf{B}\mu_n)]$. That is, not a morphism of $G_{K|k}$-actions.

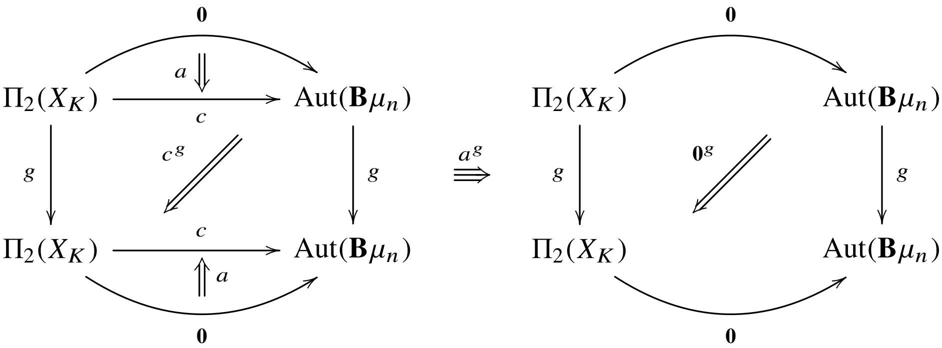
21.6. The goal is to show that in fact, a can be extended to a morphism of $G_{K|k}$-actions, that is, to a homotopy $\mathbf{0}_X \to c$. Unlike in the case of an action of a group on sets, being $G_{K|k}$-equivariant is not just a property, but also extra structure. Once this is proved, c will represent the trivial μ_n-gerbe over k, and therefore the μ_n-gerbe c has a global section over X, which is what we need to prove that X has a rational point.

Goal: *c_K is equivalent to the trivial morphism in the category of $G_{K|k}$-actions.*

21.7. The structure required for a to extend to an equivalence of $G_{K|k}$-morphisms is as follows. For each g in $G_{K|k}$, there is an equivalence

$$\begin{array}{ccc} \Pi_2(X_K) & \xrightarrow{\;c\;} & \mathrm{Aut}(\mathbf{B}\mu_n) \\ {\scriptstyle g}\big\downarrow & \overset{c^g}{\swarrow\!\!\!\!\swarrow} & \big\downarrow{\scriptstyle g} \\ \Pi_2(X_K) & \xrightarrow{\;c\;} & \mathrm{Aut}(\mathbf{B}\mu_n) \end{array}$$

giving c the structure of a $G_{K|k}$-morphism. For $a \colon \mathbf{0}_{X_K} \to c$ to be a $G_{K|k}$-2-morphism, we need for all g in $G_{K|k}$ the existence of some a^g:



where $\mathbf{0}^g$ is the 2-morphism making $\mathbf{0}\colon \Pi_2(X_K) \to \mathrm{Aut}(\mathbf{B}\mu_n)$ a morphism of $G_{K|k}$-actions.

21.8. Let $\mathcal{F}$ be the locally constant stack (that is a μ_n-gerbe) on $X_{\text{ét}}$ that c represents. Let $\mathcal{F}_K$ denote the restriction of this stack to $X_K \to X$. Then $g^{-1} \circ c \circ g$ represents $g^*\mathcal{F}$, and c^g represents the canonical equivalence between $g^*\mathcal{F}_K$ and $\mathcal{F}_K$. Then a represents a section of $\mathcal{F}$ over X_K (it represents that section which induces the equivalence of $\mathcal{F}_K$ with $\mathrm{Tors}(\mu_n)$, namely $(O_{X_K}(1), O_{X_K}^{\otimes n} \simeq L_K)$). Thus a^g represents an isomorphism of this section and its image under the action of g:

$$g^* O_{X_K} \to O_{X_K}.$$

Because $\mathrm{Pic}(X_K) \simeq \mathbf{Z}$ and the action by the Galois group fixes the degree of a line-bundle, we know that such isomorphisms exist, and furthermore, that when they exist the set of such isomorphisms is a torsor under μ_n. The difficulty therefore lies in picking a suitable a^g so that the system $(a^g \mid g \in G_{K|k})$ fits the requirements for a being $G_{K|k}$-equivariant.

21.9. The assumption of the theorem provides a weak section

$$\mathbf{B}G_k \to \Pi_2(X).$$

By lemma 14.4, this is equivalent to a section

$$s\colon \mathbf{B}G_K \to \Pi_2(X_K)$$

that is furthermore a morphism of $G_{K|k}$-actions. The assumption that the section is pseudo-geometric implies that the composite $G_{K|k}$-morphism

$$\mathbf{B}G_K \xrightarrow{s} \Pi_2(X_K) \xrightarrow{c} \mathrm{Aut}(\mathbf{B}\mu_n)$$

represents the modulo n Chern class of a line bundle on k. Since all these are trivial by an extension of Hilbert's theorem 90, we conclude that the composite $c \circ s$ is equivalent to the trivial morphism

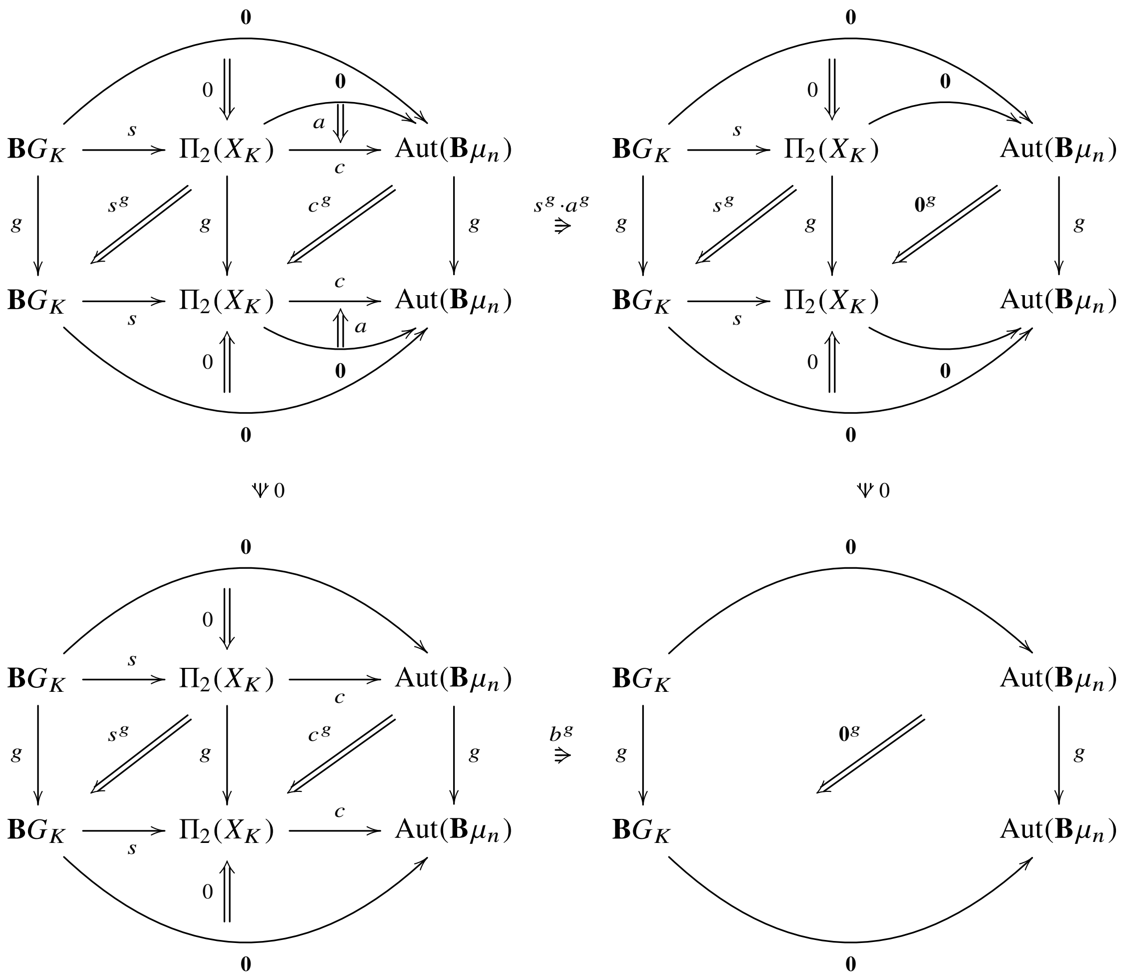
$$\mathbf{0}_k : \mathbf{B}G_K \to \mathrm{Aut}(\mathbf{B}\mu_n).$$

Denote this homotopy by b:

$$b \colon c \circ s \to \mathbf{0}_k.$$

Unlike the case for a, this 2-morphism b *is* a morphism of $G_{K|k}$-action morphisms, and it is because of this that we will be able to use b to "normalise" the various a^g to obtain a compatible system. In other words, we will extend a to a 2-morphism of $G_{K|k}$-actions such that, when pre-composed with s, we get b.

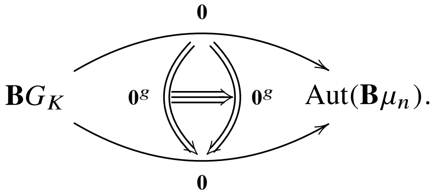
21.10. Let g be an element of $G_{K|k}$ and consider the following diagram, where a^g is a variable:



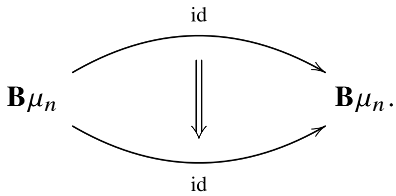
In this diagram, the morphisms labelled 0 denote equivalences of morphisms with the trivial morphism when composed with another trivial morphism.

21.11. *There exists an a^g such that the above diagram commutes.*

Indeed for any a^g, plugging it into the above diagram gives two morphisms (going opposite ways round the diagrams). These morphisms are of type:



And therefore, they are equivalent to morphisms

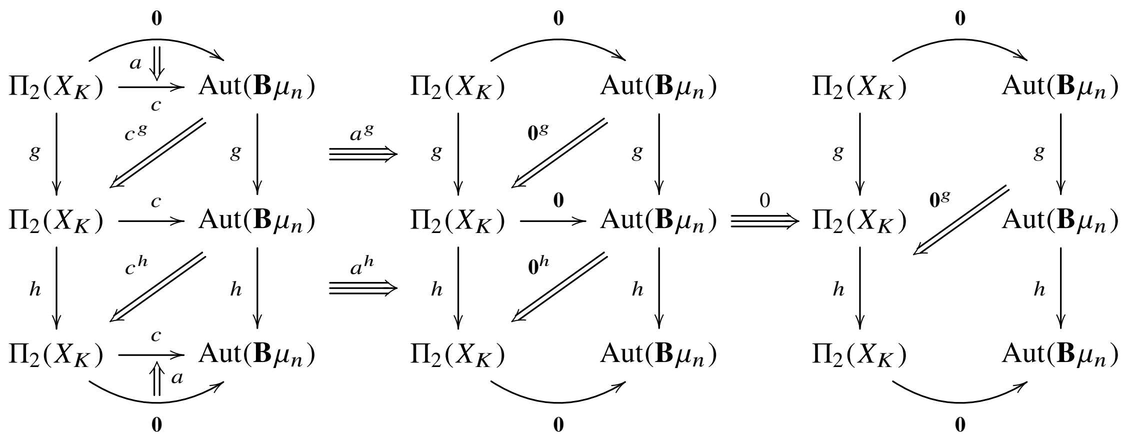


And two such morphisms differ by an element of μ_n. We pick that a^g (recall that they are a torsor under μ_n) such that the result is $0 \in \mu_n$.

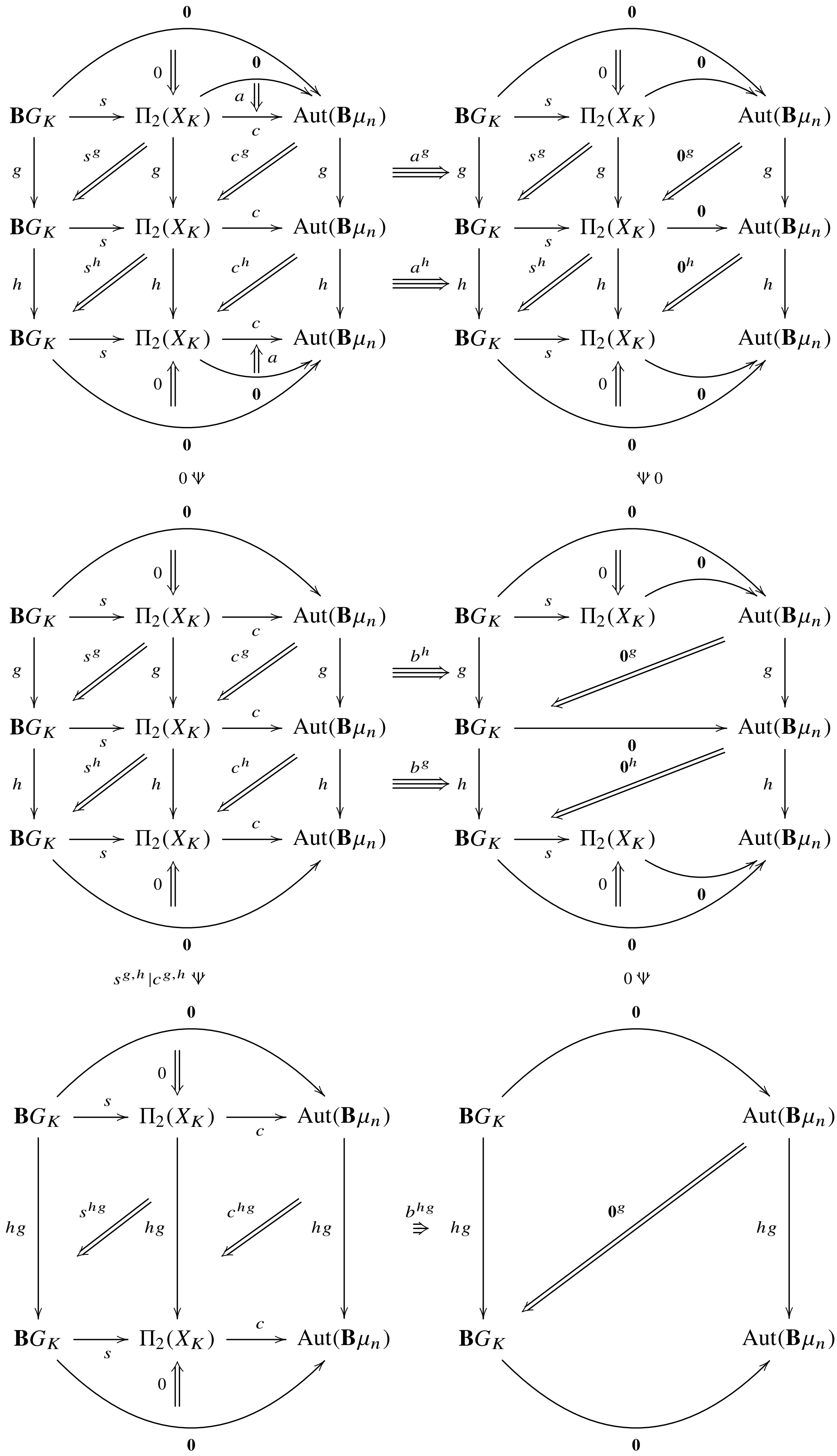
21.12. The end result is that a^g is the unique 3-morphism (with prescribed source and target) with the property that the diagram of 21.10 commutes.

21.13. Now that we have a candidate data for a $G_{K|k}$-structure given by the system $(a^g \mid g \in G_{K|k})$, we need to check that it satisfies the various requirements to make $a \colon \mathbf{0} \to c$ a homotopy in the category of $G_{K|k}$-actions.

21.14. Let g and h be elements of $G_{K|k}$. We must check that the composite



agrees with a^{hg}. (We will be suppressing the equivalences $h \circ g \to hg$ in these diagrams.) However it is clear that the top square of the diagram on the next page commutes (because the diagram in 21.10 commutes for g and h), and the bottom square commutes because $b\colon c \circ s \to \mathbf{0}$ is an equivalence in the category of $G_{K|k}$-actions.


$\mathbf{B}G_K$
$\Pi_2(X_K)$
$\mathrm{Aut}(\mathbf{B}\mu_n)$
a^g
a^h
$0\,\Psi$
$\Psi\,0$
b^h
b^g
$s^{g,h}|c^{g,h}\,\Psi$
$0\,\Psi$
b^{hg}
$\mathbf{0}^g$
$\mathbf{0}^h$
s^{hg}
c^{hg}

Therefore, the outer rectangle also commutes. Therefore, the parallel composition of a^g and a^h has the required property (that it makes the outer-rectangle commute) of a^{hg}.

21.15. This concludes the proof that a does indeed lift to a morphism in $[\mathbf{B}G_K, \mathrm{Aut}(\mathbf{B}\mu_n) \times \mathbf{B}G_K]^{G_{K|k}}_{/\mathbf{B}G_K}$, and hence that c is equivalent to the trivial μ_n-gerbe over X. Thus $O_{X_K}(1)$ is rational, and so X has a rational point. □

22. Comparison with other work

22.1. J. Schmidt obtains a similar result to 21.1 in [22], where he uses the Étale homotopy type, essentially equivalent to that of Artin and Mazur in [13], or Friedlander in [21]. His theorem 4.5.11 is:

22.2. Theorem. *Let k be a field of characteristic* 0 *and X a Brauer-Severi variety over k of period d admitting a homotopy rational point resp. a homotopy or (quasi) homology fixed point s resp. $\bar{s}$. Suppose that the class $s^*\hat{c}_1[\mathcal{L}_X]$ resp. $\bar{s}^*\hat{c}_1[LX]$ is divisible by d in $H^2(\Gamma; \hat{\mathbf{Z}}(1))$ for $[\mathcal{L}_X]$ the (positive degree) generator of* Pic(X)*. Then X is isomorphic over k to a projective space $\mathbf{P}^n$, i.e., X admits a k-rational point.*

22.3. First of all Schmidt's result uses the full étale homotopy type, whereas Theorem 21.1 only uses the homotopy 2-type, and a splitting of $\Pi_2(X) \to \mathbf{B}G_k$ is a weaker requirement than a splitting to the full homotopy type.

22.4. Schmidt's homotopy rational points are essentially the same as weak sections. His condition on the divisibility of $s^*\hat{c}_1[\mathcal{L}_X]$ by d in $H^2(\Gamma, \hat{\mathbf{Z}}(1))$ is also equivalent to the condition in Theorem 21.1 that s is pseudo-geometric, that is, that the composition with the module n Chern class of L with s is trivial.

22.5. We would like to talk about the form of both theorems from a computational point of view. Theorem 22.2 uses the étale homotopy type, which is an object in the homotopy category of pro-simplicial sets (or a pro-object in the category of simplicial sets).

Simplicial sets are a model for homotopy theory that have produced good results from a theoretical point of view. In particular, they can immediately handle higher-homotopical data without having to elaborate on higher coherence conditions. They have been put to good use in the theory of $(\infty, 1)$-Topoi, for example. However because simplicial sets are in essence non-algebraic, they are a form of data that is hard to handle algorithmically.

Because 2-groupoids have well defined composition operators, one can present 2-groupoids in much the same way one presents finitely generated groups: using 0-generators, 1-generators, 2-generators, and relations. The same methods which are used to perform computations in current computer algebra software packages can be applied to higher-groupoids. The software package SAGE [sage] for example can only deal with *finite* simplicial complexes, whereas it is straightforward to write down the generators and relations for $\mathbf{B}^n A$, where A is a finite group, and to perform computations using the generated data.

22.6. Finally we would like to comment on the nature of the proofs. Schmidt's proof relies mostly on cohomological computations, which are augmented by the use of a homotopy section, which provides maps of cohomology groups. In our setting we also get étale cohomology: the equivalence classes of morphisms

$$\Pi_2(X) \to \mathbf{B}^2\mu_n /\!\!/ G_k \quad \text{over } \mathbf{B}G_k$$

is in bijection with $H^2(X, \mu_n)$ (where $\mathbf{B}^2\mu_n /\!\!/ G_k$ is the weak quotient of $\mathbf{B}\mu_n$ by the action of G_k). However we prefer to deal with the morphisms directly, as we feel this more closely resembles the geometry of the underlying problem: we are using the section s, and its structure as a homotopy fixed point of $\Pi_2(X_K)$ under the action of $G_{K|k}$, to normalise descent data for the morphism that represents the Chern class. In our opinion this gives a clearer and simplified proof of the theorem, in which in particular, the condition on s (being pseudo-geometric) seems less arbitrary.

Part 9. Further work

23. Locally ∞-ringed ∞-groupoids

In this part we discuss how the work presented here and in particular the notion of a pseudo-geometric morphism, might fit into a larger theory. We will freely use the notion of an algebraic notion of n-groupoids, and ∞-groupoids, leaving the foundations unspecified. One possibility would also be to frame all of this in Homotopy type theory.

23.1. The basic structure of a scheme X, as given by Grothendieck, is an underlying topological space X_t that is equipped with a sheaf of rings $\mathcal{O}_X$ on X_t. Such an object is a scheme, if it is furthermore locally an affine scheme, that is, isomorphic to $\mathrm{Spec}(A)$ for some ring A. This then makes X into a locally ringed space. A morphism of schemes $f\colon X \to Y$ is then a morphism of locally ringed spaces, which in particular specifies a morphism of topological spaces

$$f_t\colon X_t \to Y_t$$

and a morphism of sheaves of rings

$$f_{\mathcal{O}}\colon \mathcal{O}_Y \to f_{t*}\mathcal{O}_X.$$

23.2. There is a "theory of rings", which can simply be defined as the opposite of the category of finitely generated free $\mathbf{Z}$-algebras. That is, the opposite $\mathcal{R}$ of the category:

{0} for each integer $n \geqslant 0$, an object R_n which is just the polynomial algebra $\mathbf{Z}[x_1, \ldots, x_n]$,
{1} As morphisms, the morphisms of rings.

The category $\mathcal{R}$ is a Lawvere theory (see Chapter 4 of [24]). It has finite products,

$$R_n \times R_m \simeq R_{n+m}.$$

A *model* for the theory $\mathcal{R}$ is a product preserving functor

$$A\colon \mathcal{R} \to \mathbf{Set}.$$

Such a functor produces an actual ring in the classical sense. The object R_1 gets mapped to a set $A(R_1)$, and for example the multiplication of the ring is given by

$$A(R_1 \times R_1) \simeq A(R_1) \times A(R_1) \to A(R_1)$$

the image of (the opposite of):

$$\mathbf{Z}[z] \to \mathbf{Z}[x, y]$$

$$z \mapsto xy$$

In essence the category $\mathcal{R}$ encodes, without bias towards $0, 1, xy, x + y$, all the polynomial functions that iterated products of a set should be equipped with, and how they compose with one another.

23.3. The theory $\mathcal{R}$ can be used to define rings in other categories, and in particular, into higher categories. Thus, a ∞-*groupoid ring* is defined to be a (weak) product-preserving functor

$$\mathcal{R} \to \infty\mathbf{Gpd}.$$

Thus an ∞-groupoid ring is an infinity groupoid M that is equipped with addition and multiplication morphisms, 0 and 1 objects, but also higher morphisms that encode weaker notions of distributivity, etc. The category of ∞-groupoid rings will be denoted $\mathcal{R}(\infty\mathbf{Gpd})$.

23.4. Example. Consider a field k, and an algebraic closure $\bar{k}$ of k. Then $\bar{k}$ is a ring, and it has an an action by the group G_k, the absolute Galois group of k. Considering $\bar{k}$ as an ∞-groupoid that happens to be a homotopy 0-type, we have an ∞-groupoid ring $\bar{k}$ with an action by G_k. The action by G_k is given by the data of a functor

$$\rho\colon \mathbf{B}G_k \to \infty\mathbf{Gpd}$$

such that $\rho(*) = \bar{k}$, and each $g \in G_k$, that is, each morphism $g\colon * \to *$ gets sent to the action $g\colon \bar{k} \to \bar{k}$. Because the action by G_k actually preserves the ring structure, we actually have a functor

$$\mathbf{B}G_k \to \mathcal{R}(\infty\mathbf{Gpd}).$$

Define $\bar{k} /\!\!/ G_k$ to be the (weak) colimit of ρ:

$$\bar{k} /\!\!/ G_k := \varinjlim(\rho).$$

Because the action by G_k on $\bar{k}$ is an action in the category of rings, the resulting ∞-groupoid $\bar{k} /\!\!/ G_k$ is actually an ∞-groupoid ring. By the universal property of $\varinjlim(\rho)$, there is a morphism

$$\bar{k} /\!\!/ G_k \to \mathbf{B}G_k.$$

23.5. In the example just given, we have an ∞-groupoid ring $\bar{k} /\!\!/ G_k$ over $\Pi_1(\mathrm{Spec}(k)) \simeq \Pi_\infty(\mathrm{Spec}(k))$. The sheaf of rings of the scheme $\mathrm{Spec}(k)$ is just k, but note that the decategorification of $\bar{k} /\!\!/ G_k$ is isomorphic to k, that is, if the colimit of $\bar{k}$ under the action of G_k is taken in sets rather than ∞-groupoids, then we obtain k.

Thus the morphism

$$\bar{k} /\!\!/ G_k \to \Pi_\infty(\mathrm{Spec}(k))$$

can be seen as a more refined analogue of the sheaf of rings on the underlying topological space of $\mathrm{Spec}(k)$. The underlying topological space of $\mathrm{Spec}(k)$, that is, the Zariski topology, which is clearly somewhat lacking in this case (it is just a point), has been replaced by an object representing its étale homotopy type.

23.6. We speculate that there is a way to construct a rigidified étale homotopy type $\bar{\Pi}_\infty(X)$ of a scheme, in such a way that the sheaf of rings of X can be encoded by a morphism

$$\mathcal{O}_X^\infty \to \bar{\Pi}_\infty(X),$$

where $\mathcal{O}_X^\infty$ is an ∞-groupoid ring. The example of $\bar{k} /\!\!/ G_k \to \mathbf{B}G_k$ would then be a special case of this, where $\bar{\Pi}_\infty(\mathrm{Spec}(k)) \simeq \mathbf{B}G_k$.

The original sheaf of rings would be recovered as follows. The global sections $\mathcal{O}(X)$ correspond to equivalence classes of sections of $\mathcal{O}_X^\infty \to \bar{\Pi}_\infty(X)$. For an étale morphism $u\colon U \to X$, one obtains

$$\bar{\Pi}_\infty(u)\colon \bar{\Pi}_\infty(U) \to \bar{\Pi}_\infty(X)$$

and the sections of $\mathcal{O}_X$ over U are the equivalence classes of sections a: that make the following diagram commute:

$$\begin{array}{ccc} & & \mathcal{O}_X^\infty \\ & \overset{a}{\nearrow} & \downarrow \\ \bar{\Pi}_\infty(U) & \longrightarrow & \bar{\Pi}_\infty(X). \end{array}$$

Because the sections a land in an ∞-groupoid ring, the set equivalence classes naturally forms a ring.

23.7. We can check that this works for $\bar{k} /\!\!/ G_k \to \mathbf{B}G_k$. The sections of $\bar{k} /\!\!/ G_k \to \mathbf{B}G_k$ are homotopy fixed points of $\bar{k}$ under the action of G_k, that is, elements of k. And for an étale extension of fields $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$, the equivalence classes of sections a:

$$\begin{array}{ccc} & & \bar{k} /\!\!/ G_k \\ & \overset{a}{\nearrow} & \downarrow \\ \mathbf{B}G_K & \longrightarrow & \mathbf{B}G_k. \end{array}$$

correspond to G_K-invariants, that is, elements of K.

24. The sections conjecture

24.1. The problem with a generalised sections conjecture, that is, trying to classify rational point $X(k)$ by equivalence classes sections of $\Pi_\infty(X) \to \mathbf{B}G_k$, is that too much geometry is forgotten. In the case of Brauer-Severi varieties above, this can be remedied by requiring that some geometry is preserved; this is what gives rise to the notion of a pseudo-geometric morphism in Definition 17.3. By adding the structure of $\mathcal{O}_X$ to $\bar{\Pi}_\infty(X)$, all the geometry can be preserved. We can define a morphism

$$(\mathcal{O}_X^\infty \to \bar{\Pi}_\infty(X)) \to (\mathcal{O}_Y^\infty \to \bar{\Pi}_\infty(Y))$$

to ba morphism

$$f\colon \bar{\Pi}_\infty(X) \to \bar{\Pi}_\infty(Y)$$

and a morphism

$$\varphi\colon f^{-1}\mathcal{O}_Y^\infty \to \mathcal{O}_X.$$

over $\Pi_\infty(X)$, together with a suitable analogue for "locally ringed".

24.2. Example. Consider the case of the affine scheme $\mathrm{Spec}(k[x]) = \mathbf{A}_k^1$ over a field k. Assume for the moment that $\bar{k}$ is algebraically closed. Then we obtain

$$k \to \mathbf{1}$$

for $\mathrm{Spec}(k)$ and because $\Pi_\infty(\mathbf{A}_k^1) \simeq 1$, we have the presentation

$$k[x] \to \mathbf{1}$$

for $\mathbf{A}_k^1$. When k is not algebraically closed, these schemes produce:

$$\bar{k} /\!/ G_k \to \mathbf{B}G_k$$

$$\bar{k}[x] /\!/ G_k \to \mathbf{B}G_k$$

Thus the k-points of $\mathbf{A}_k^1$ would be computed by the equivalence classes of sections a:

$$\begin{array}{ccc} \bar{k}[x] /\!/ G_k & \xrightarrow{\quad a \quad} & \bar{k} /\!/ G_k \\ & \searrow \quad \swarrow & \\ & \mathbf{B}G_k. & \end{array}$$

These are equivalent to G_k-equivariant morphisms $\bar{k}[x] \to \bar{k}$, and thus are fully determined by the G_k-stable image of x, an element of k. Thus, in this case the morphisms do compute the rational points of $\mathbf{A}_k^1$.

24.3. In this case it would be reasonable to expect that equivalence classes of such morphisms classify exactly the morphisms of schemes. This could give a new way to attack the Grothendieck sections conjecture. For an anabelian scheme X, the rational points of X would correspond to morphisms

$$f\colon \mathbf{B}G_k \to \bar{\Pi}_\infty(X)$$

equipped with

$$\varphi\colon f^{-1}\mathcal{O}_X \to \bar{k} /\!\!/ G_k$$

over $\mathbf{B}G_k$. Since Anabelian varieties are homotopy 1-types, f would be simplified to a morphism $f\colon \mathbf{B}G_k \to \bar{\Pi}_1(X)$, which is already what is studied by the sections conjecture. The content of the sections conjecture would then simply be that given any f, it is possible to construct such a φ, and that this φ is furthermore uniquely determined up to equivalence. The point is that the main difficulty in proving a sections conjecture type theorem is getting at the existence of a point. This gives a clear technique for proving that a point exists.

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